Instantons in $\mathcal{N} = 2$ magnetized D–brane worlds

Marco Billò, Marialuisa Frau, Igor Pesando

Dipartimento di Fisica Teorica, Università di Torino
and I.N.F.N. - sezione di Torino
Via P. Giuria 1, I-10125 Torino, Italy

Paolo Di Vecchia

The Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark
and NORDITA, Roslagstullsbacken 23, SE-10691 Stockholm, Sweden

Alberto Lerda

Dipartimento di Scienze e Tecnologie Avanzate, Università del Piemonte Orientale
and I.N.F.N. - Gruppo Collegato di Alessandria - sezione di Torino
Via V. Bellini 25/G, I-15100 Alessandria, Italy

Raffaele Marotta

I.N.F.N. - sezione di Napoli and Dipartimento di Scienze Fisiche, Università di Napoli.
Complesso Universitario Monte S. Angelo - ed. G
Via Cintia - I-80126 Napoli, Italy

ABSTRACT: In a toroidal orbifold of type IIB string theory we study instanton effects in $\mathcal{N} = 2$ super Yang-Mills theories engineered with systems of wrapped magnetized D9 branes and Euclidean D5 branes. We analyze the various open string sectors in this brane system and study the 1-loop amplitudes described by annulus diagrams with mixed boundary conditions, explaining their rôle in the stringy instanton calculus. We show in particular that the non-holomorphic terms in these annulus amplitudes precisely reconstruct the appropriate Kähler metric factors that are needed to write the instanton correlators in terms of purely holomorphic variables. We also explicitly derive the correct holomorphic structure of the instanton induced low energy effective action in the Coulomb branch.

Contents

1. Introduction

In their original formulation string theories were defined as perturbative expansions in the string coupling constant g_s that reproduce the corresponding perturbative field theoretical expressions in the zero-slope limit ($\alpha' \to 0$). For a long time it seemed very difficult, or even impossible, to reproduce in string theory the non-perturbative effects that were instead known from field theory, such as for example instanton effects.

A first important step in this direction was performed in Ref. [1], but it was only after the discovery of string dualities and M theory that a real progress could be achieved. In fact, by exploiting string dualities it became clear that perturbative phenomena in one theory often correspond to non-perturbative ones in the dual theory and vice-versa, and that the dependence on the string coupling constant of these non-perturbative effects is of the same type produced by instantons in field theory [2]. Non-perturbative phenomena of this kind were discovered both in type II theories [3, 4, 5] and in the framework of Heterotic/Type I duality [6].

These developments opened the way to a more systematic analysis of instanton effects in string theory [7, 8]. Among the stringy non-perturbative configurations, the so-called D-instantons, *i.e.* the D(–1) branes of type IIB, were the mostly studied ones at the beginning and, after the discovery of the AdS/CFT correspondence, they were intensively used to get additional evidence of the equivalence between $\mathcal{N} = 4$ super Yang-Mills theory (SYM) in four dimensions and type IIB string theory on $AdS_5 \times S^5$ [9]–[13].

These results were largely based on the fact that the instanton sectors of $\mathcal{N} = 4$ SYM theory can be described in string theory by systems of D3 and D(–1) branes (or D-instantons) [14, 15, 8]. In fact, the excitations of the open strings stretching between two D(–1) branes, or between a D3 brane and a D-instanton, are in one-to-one correspondence with the moduli of the SYM instantons in the so-called ADHM construction (for comprehensive reviews on the subject see, for example, Refs. [16, 17]). This observation can be further substantiated [18] by showing that the tree-level string scattering amplitudes on disks with mixed boundary conditions for a D3/D(–1) system lead, in the $\alpha' \to 0$ limit, to the effective action on the instanton moduli space of the SYM theory. Moreover, it can be proved [18] that the same disk diagrams also yield the classical profile of the gauge instanton solution, in close analogy with the procedure that generates the profile of the classical supergravity D brane solutions from boundary states [19].

This approach can be easily adapted to describe gauge instantons in SYM theories with reduced supersymmetry by placing the D3/D(–1) systems at suitable orbifold singularities. It is also possible to take into account the deformations induced by non-trivial gravitational backgrounds both of NS-NS and R-R type [20, 21, 22]. For instance, by studying a D3/D(–1) system in an $\mathcal{N} = 2$ orbifold and in the presence of a graviphoton background it is possible to systematically obtain the instanton induced gravitational corrections to the $\mathcal{N} = 2$ low-energy effective SYM action using perturbative string methods [22].

More recently, the string description of instantons has lead to new developments that have received a lot of attention. In fact it has been shown in several different contexts [23]–[37] that the stringy instantons may dynamically generate new types of superpotential terms in the low-energy effective action of the SYM theory. These new types of F-terms may have very interesting phenomenological implications, most notably they can provide a mechanism for generating Majorana masses for neutrinos [24, 25] in some semi-realistic string extensions of the Standard Model.

However, one of the problems that one has to face in this approach is that a superpotential term must be holomorphic in the appropriate field theory variables, but what is holomorphic in string theory is not quite the same of what is holomorphic in supergravity. If we limit ourselves to a toroidal compactification of string theory of the type $\mathbb{R}^{1,3} \times \mathcal{T}_2^{(1)} \times \mathcal{T}_2^{(2)} \times \mathcal{T}_2^{(3)}$, the holomorphic quantities that naturally appear are the complex structures and the Kähler structures of the three tori, together with the ten-dimensional axion-dilaton field. On the other hand, when we incorporate the results of the string compactification in a four-dimensional supergravity Lagrangian, the appropriate fields to be used are different from those mentioned above and are obtained from these by forming specific combinations with various R-R fields (see, for instance, Ref. [38] for a review). Only when written in terms of these supergravity variables, the F-terms have the correct

holomorphic structure and the tree-level SYM coupling constant is the sum of a holomorphic and an anti-holomorphic quantity as required by supersymmetry. When 1-loop effects are included, some non-holomorphic terms appear due to the presence of massless modes which require an IR regularization procedure, but they turn out to precisely reconstruct the Kähler metrics of the various low-energy fields [39, 40] [1], so that they can be re-absorbed with field redefinitions.

A similar pattern should occur also for the non-perturbative F-terms induced by instantons in string models. While the holomorphic dependence of these instanton contributions from the complex quantities of the low-energy theory is a consequence of the cohomology properties of the integration measure on the instanton moduli space [48, 16, 22], the holomorphic dependence on the compactification moduli is not at all obvious. This problem has started to be analyzed only recently in the framework of intersecting brane worlds in type IIA string theory [33].

In this paper we consider instead a toroidal orbifold compactification of type IIB string theory in $\mathbb{R}^{1,3} \times \frac{\mathcal{T}_2^{(1)} \times \mathcal{T}_2^{(2)}}{\mathbb{Z}_2} \times \mathcal{T}_2^{(3)}$ and study systems of fractional D9 branes that are wrapped and magnetized on the three tori in such a way to engineer a $\mathcal{N} = 2$ SYM theory with N_F flavors. In particular we distinguish the color D9 branes, which support the degrees of freedom of the gauge multiplet, and the flavor D9 branes, which instead give rise to hyper-multiplets in the fundamental representation. To study instanton effects in this set-up, we add a stack of Euclidean D5 branes (E5 branes for short) that completely wrap the internal manifold and hence describe point-like configurations from the four-dimensional point of view [2]. If the wrapping numbers and magnetization of these E5 branes are the same as those of the color D9 branes, we have a stringy realization of ordinary gauge theory instantons; if instead the internal structure of the wrapped E5 branes differs from that of the color branes, then we have exotic instanton configurations of truly stringy nature. In this paper we will consider the first case, but in principle our results can be useful also to study the exotic cases. The physical excitations corresponding to open strings with at least one end-point on the E5 branes describe the instanton moduli, and their mutual interactions, as well as their couplings with the gauge and matter fields, can be explicitly obtained from the $\alpha' \to 0$ limit of disk diagrams with mixed boundary conditions, in complete analogy with the $\mathcal{N} = 2$ system studied in Ref. [22] in a non-compact orbifold. In our case, however, we have to take into account also the contribution of the compact internal space, and in particular of its complex and Kähler structure moduli which explicitly appear in the 1-loop amplitudes corresponding to annulus diagrams with one boundary on the instantonic E5 branes and the other on the D9 branes. We show with very general arguments that in supersymmetric gauge theories these annulus diagrams with mixed boundary conditions describe precisely the 1-loop correction to the gauge coupling constant, in agreement with some recent observations [26, 27]. Besides the usual logarithmic terms that are responsible for the running of the coupling constant, these 1-loop corrections in general contain also some finite terms that are interpreted as threshold effects [39, 49].

---

[1]See also Refs. [41]–[47].

[2]These D9/E5 systems are essentially a T-dual version of the D3/D(−1) systems mentioned above.

While in the non-compact orbifolds these thresholds are absent [50, 51, 52], in the non-compact case they give, instead, a relevant contribution and actually produce crucial non-holomorphic terms that precisely reconstruct the appropriate Kähler metric factors which compensate those arising in the transformation from the string to the supergravity basis. In this way one can explicitly prove that the instanton induced low-energy effective action, when written in the supergravity variables, has the correct holomorphic properties, as required by supersymmetry.

The paper is organized as follows. In Section 2 we review how to engineer $\mathcal{N} = 2$ SYM theories with flavors using wrapped magnetized D9 branes in a toroidal orbifold compactification of type II string theory and discuss the relation between the string basis and the supergravity basis which allows to determine the form of the Kähler metric for the various scalar fields of the model. In Section 3 we describe the instanton calculus in string theory and discuss how to obtain the instanton induced contributions to the low-energy effective action from disk amplitudes. We also show how the 1-loop annulus amplitudes enter in the calculation. Section 4 is devoted to perform the explicit computation of these annulus amplitudes and to explain their rôle in the instanton calculus. In Section 5 we show that the non-perturbative effective actions generated by the E5 branes have the correct holomorphic structure required by supersymmetry for Wilsonian actions, if the appropriate variables of the supergravity basis are used. Finally in Section 6 we present our conclusions and in the Appendix we provide some technical details for the integral appearing in the annulus amplitudes.

2. $\mathcal{N} = 2$ models from magnetized branes

In this section we review how to obtain gauge theories with $\mathcal{N} = 2$ supersymmetry from systems of magnetized D9 branes in a toroidal orbifold compactification of Type IIB string theory.

To set our notations, let us first give some details on the background geometry. We take the space-time to be the product of $\mathbb{R}^{1,3}$ times a six-dimensional factorized torus $\mathcal{T}_6 = \mathcal{T}_2^{(1)} \times \mathcal{T}_2^{(2)} \times \mathcal{T}_2^{(3)}$. For each torus $\mathcal{T}_2^{(i)}$, the string frame metric and the B-field are parameterized by the Kähler and complex structure moduli, respectively $T^{(i)} = T_1^{(i)} + \mathrm{i}\, T_2^{(i)}$ and $U^{(i)} = U_1^{(i)} + \mathrm{i}\, U_2^{(i)}$, according to

$$G^{(i)} = \frac{T_2^{(i)}}{U_2^{(i)}} \begin{pmatrix} 1 & U_1^{(i)} \\ U_1^{(i)} & |U^{(i)}|^2 \end{pmatrix} \quad \text{and} \quad B^{(i)} = \begin{pmatrix} 0 & -T_1^{(i)} \\ T_1^{(i)} & 0 \end{pmatrix} . \tag{2.1}$$

In our conventions, the dimensionful volume of the i-th torus is $(2\pi\sqrt{\alpha'})^2 T_2^{(i)}$. This toroidal geometry breaks $\mathrm{SO}(1,9)$ into $\mathrm{SO}(1,3) \times \prod_i \mathrm{U}(1)^{(i)}$, and correspondingly the ten-dimensional string coordinates X^M and ψ^M are split as

$$X^M \to (X^\mu, Z^i) \quad \text{and} \quad \psi^M \to (\psi^\mu, \Psi^i) \tag{2.2}$$

where $\mu = 0, 1, 2, 3$ and [3]

$$Z^i = \sqrt{\frac{T_2^{(i)}}{2U_2^{(i)}}} \left(X^{2i+2} + U^{(i)} X^{2i+3} \right) \quad , \quad \Psi^i = \sqrt{\frac{T_2^{(i)}}{2U_2^{(i)}}} \left(\psi^{2i+2} + U^{(i)} \psi^{2i+3} \right) \tag{2.3}$$

for $i = 1, 2, 3$. Similarly, the (anti-chiral)[4] spin-fields $S^{\dot{\mathcal{A}}}$ of the RNS formalism in ten dimensions factorize in a product of four-dimensional and internal spin-fields according to

$$S^{\dot{\mathcal{A}}} \to (S_\alpha S_{---}, S_\alpha S_{-++}, S_\alpha S_{+-+}, S_\alpha S_{++-}, S^{\dot{\alpha}} S^{+++}, S^{\dot{\alpha}} S^{+--}, S^{\dot{\alpha}} S^{-+-}, S^{\dot{\alpha}} S^{--+}) \tag{2.4}$$

where the index α ($\dot{\alpha}$) denotes positive (negative) chirality in $\mathbb{R}^{1,3}$ and the labels $(\pm, \pm, \pm)$ on the internal spin-fields denote charges $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$ under the three internal U(1)'s.

Without loss of generality, we set the B-field to zero (at the end of this section we will see how to incorporate it). The above geometry can also be described in the so-called supergravity basis using the complex moduli s, $t^{(i)}$ and $u^{(i)}$, whose relation with the previously introduced quantities in the string basis is (see for instance Ref. [53, 38])

$$\begin{aligned}
\mathrm{Im}(s) &\equiv s_2 = \frac{1}{4\pi} \, \mathrm{e}^{-\phi_{10}} \, T_2^{(1)} T_2^{(2)} T_2^{(3)} \ , \\
\mathrm{Im}(t^{(i)}) &\equiv t_2^{(i)} = \mathrm{e}^{-\phi_{10}} T_2^{(i)} \ , \\
u^{(i)} &= u_1^{(i)} + \mathrm{i}\, u_2^{(i)} = U^{(i)} \ ,
\end{aligned} \tag{2.5}$$

where ϕ_{10} is the ten dimensional dilaton. The real parts of s and $t^{(i)}$ are related to suitable R-R potentials. In terms of these variables, the bulk Kähler potential in the $\mathcal{N} = 1$ language[5] is given by [54]

$$K = -\log(s_2) - \sum_{i=1}^{3} \log(t_2^{(i)}) - \sum_{i=1}^{3} \log(u_2^{(i)}) \ . \tag{2.6}$$

2.1 The gauge sector

In the above toroidal background we now introduce a stack of N_a D9 branes. The open string excitations that are massless in $\mathbb{R}^{1,3}$ describe a Super Yang-Mills (SYM) theory with gauge group $U(N_a)$ and $\mathcal{N} = 4$ supersymmetry in four dimensions. In order to reduce to $\mathcal{N} = 2$, we replace $\mathcal{T}_6$ with the toroidal orbifold

$$\frac{\mathcal{T}_2^{(1)} \times \mathcal{T}_2^{(2)}}{\mathbb{Z}_2} \times \mathcal{T}_2^{(3)} \ , \tag{2.7}$$

---

[3]The prefactors in (2.3) are chosen in such a way that the complex coordinates are orthonormal in the metric (2.1).

[4]We define the 10-dimensional GSO projection so that in the Ramond zero-mode sector it selects anti-chiral states; in other words, in this sector we take $(-1)^F$ to be given by *minus* the chirality matrix Γ_{11}.

[5]Strictly speaking this Kähler potential is not globally defined since the scalars of the hypermultiplets $T^{(1)}, T^{(2)}$ and $U^{(1)}, U^{(2)}$ live in a quaternionic manifold, which is not Kähler since its holonomy group is not contained in $U(n)$. The quaternionic manifold of $N = 2$ supergravity becomes an hyperKähler manifold of $N = 2$ rigid supersymmetry in the limit where the gravitational interaction is switched off: the Kähler potential we use in this work has therefore to be interpreted as the expression one obtains in the rigid limit or as a local expression.

where $\mathbb{Z}_2$ simply acts as a reflection in the first two tori (*i.e.* $Z^i \to -Z^i$ for $i = 1, 2$), and consider fractional D9 branes instead of bulk branes[6]. Actually, in the orbifold (2.7) there are two types of fractional branes corresponding to the two irreducible representations of $\mathbb{Z}_2$ that can be assigned to the open string Chan-Paton factors. For simplicity, we take all the N_a D9 branes to be fractional branes of the same kind (for example with the trivial representation on the Chan-Paton factors) and we will call them color branes. Then, one can easily see that the physical massless open string states surviving the orbifold projection are a vector A_μ, a complex scalar ϕ and two gaugini $\Lambda^{\alpha 1}$ and $\Lambda^{\alpha 2}$. They are described by the following vertex operators

$$V_A(z) = (\pi\alpha')^{\frac{1}{2}} A_\mu \, \psi^\mu(z) \, \mathrm{e}^{-\varphi(z)} \, \mathrm{e}^{\mathrm{i}p_\mu X^\mu(z)} \, , \tag{2.8a}$$

$$V_\phi(z) = (\pi\alpha')^{\frac{1}{2}} \phi \, \Psi^3(z) \, \mathrm{e}^{-\varphi(z)} \, \mathrm{e}^{\mathrm{i}p_\mu X^\mu(z)} \tag{2.8b}$$

in the (-1) superghost picture of the NS sector, and

$$V_{\Lambda^1}(z) = (2\pi\alpha')^{\frac{3}{4}} \Lambda^{\alpha 1} \, S_\alpha(z) S_{+-+}(z) \, \mathrm{e}^{-\frac{1}{2}\varphi(z)} \, \mathrm{e}^{\mathrm{i}p_\mu X^\mu(z)} \, , \tag{2.9a}$$

$$V_{\Lambda^2}(z) = (2\pi\alpha')^{\frac{3}{4}} \Lambda^{\alpha 2} \, S_\alpha(z) S_{-++}(z) \, \mathrm{e}^{-\frac{1}{2}\varphi(z)} \, \mathrm{e}^{\mathrm{i}p_\mu X^\mu(z)} \tag{2.9b}$$

in the $(-1/2)$ superghost picture of the R sector. We have defined the action of the $\mathbb{Z}_2$ orbifold generator h on the R ground states to be

$$h = -\sigma_3 \otimes \sigma_3 \otimes 1 \, , \tag{2.10}$$

which is the spinor representation of a π rotation in the first two tori. Then, one can easily see that the two internal spin fields in the fermionic vertices (2.9) have h-parity one and are selected by the orbifold projection

$$P_{\mathrm{orb}} = \frac{1+h}{2} \, . \tag{2.11}$$

In all vertices (2.8) and (2.9), the polarizations have canonical dimensions (this explains the dimensional prefactors [7]) and are $N_a \times N_a$ matrices transforming in the adjoint representation of $\mathrm{SU}(N_a)$ (here we neglect an overall factor of $\mathrm{U}(1)$, associated to the center of mass of the N_a D9 branes, which decouples and does not play any rôle in our present context). The vertex operators (2.8) and (2.9) describe the components of a $\mathcal{N} = 2$ vector superfield and are connected to each other by the following supercharges:

$$Q_{\alpha 1} = \oint \frac{dz}{2\pi\mathrm{i}} S_\alpha(z) S_{-++}(z) \, \mathrm{e}^{-\frac{1}{2}\varphi(z)} \, , \quad Q_{\alpha 2} = \oint \frac{dz}{2\pi\mathrm{i}} S_\alpha(z) S_{+-+}(z) \, \mathrm{e}^{-\frac{1}{2}\varphi(z)} \, ,$$

$$\bar{Q}^{\dot\alpha 1} = \oint \frac{dz}{2\pi\mathrm{i}} S^{\dot\alpha}(z) S^{+--}(z) \, \mathrm{e}^{-\frac{1}{2}\varphi(z)} \, , \quad \bar{Q}^{\dot\alpha 2} = \oint \frac{dz}{2\pi\mathrm{i}} S^{\dot\alpha}(z) S^{-+-}(z) \, \mathrm{e}^{-\frac{1}{2}\varphi(z)} \, , \tag{2.12}$$

which generate the $\mathcal{N} = 2$ supersymmetry algebra selected by the orbifold projection (2.11).

---

[6]The twisted closed string sectors introduced by the orbifold will not play any rôle for our considerations, and thus it is enough to still consider only the untwisted moduli (2.5).

[7]See for example Ref. [18] for details on the normalizations of vertex operators and scattering amplitudes.

By computing all tree-level scattering amplitudes among the vertex operators (2.8) and (2.9) and their conjugates, and taking the field theory limit $\alpha' \to 0$, one can obtain the $\mathcal{N} = 2$ SYM action

$$
\begin{aligned}
S_{\text{SYM}} = \frac{1}{g_a^2} \int d^4x \, \text{Tr} \Big\{ & \frac{1}{2} F_{\mu\nu}^2 + 2 \, D_\mu \bar{\phi} \, D^\mu \phi - 2 \, \bar{\Lambda}_{\dot{\alpha}A} \bar{\not{D}}^{\dot{\alpha}\beta} \Lambda_\beta^A \\
& + \mathrm{i}\sqrt{2} \, \bar{\Lambda}_{\dot{\alpha}A} \epsilon^{AB} \big[\phi, \bar{\Lambda}_B^{\dot{\alpha}} \big] + \mathrm{i}\sqrt{2} \, \Lambda^{\alpha A} \epsilon_{AB} \big[\bar{\phi}, \Lambda_\alpha^B \big] + \big[\phi, \bar{\phi} \big]^2 \Big\} ,
\end{aligned}
\tag{2.13}
$$

where $A, B = 1, 2$, and the Yang-Mills coupling constant g_a is given by

$$
\frac{1}{g_a^2} = \frac{1}{4\pi} \, \mathrm{e}^{-\phi_{10}} \, T_2^{(1)} T_2^{(2)} T_2^{(3)} = s_2 .
\tag{2.14}
$$

Since we will study instanton effects, we have written the above action with Euclidean signature.

For later convenience it is useful to compare the bosonic part of the action (2.13) with the Dirac-Born-Infeld (DBI) action for D9 branes in the toroidal orbifold (2.7). In the Euclidean string frame, this action is given by

$$
S_{\text{DBI}} = \frac{2\pi}{(2\pi\sqrt{\alpha'})^{10}} \int d^{10}x \, \mathrm{e}^{-\phi_{10}} \sqrt{\det \left(G_{MN} + 2\pi\alpha' F_{MN} \right)} ,
\tag{2.15}
$$

where G_{MN} is the world-volume metric and F_{MN} is a gauge field strength. Promoting the latter to be non-abelian [8], and compactifying S_{DBI} to four dimensions on the toroidal orbifold (2.7), the quadratic terms in F read (see also Refs. [55, 52])

$$
\int d^4x \, \sqrt{\det G_4} \, \text{Tr} \Big\{ \frac{1}{2g_a^2} F_{\mu\nu}^2 + 2 \, \mathrm{e}^{-\phi_{10}} \prod_{i=1}^{3} \sqrt{\det G^{(i)}} \frac{1}{T_2^{(3)} U_2^{(3)}} D_\mu \bar{\Phi} D^\mu \Phi \Big\} ,
\tag{2.16}
$$

where G_4 is the string frame metric in the non-compact space and

$$
\Phi = \frac{1}{\sqrt{4\pi}} \left(U^{(3)} A_8 - A_9 \right)
\tag{2.17}
$$

with A_8 and A_9 denoting the components of the ten-dimensional gauge field along $\mathcal{T}_2^{(3)}$. Changing to the (flat) Euclidean Einstein frame with

$$
(G_4)_{\mu\nu} = \mathrm{e}^{2\phi_4} \, \delta_{\mu\nu} ,
\tag{2.18}
$$

where $\phi_4 = \phi_{10} - \frac{1}{2} \sum_i \log(T_2^{(i)})$ is the four-dimensional dilaton, and using the geometrical moduli (2.5) of the supergravity basis, we can rewrite (2.16) as

$$
\int d^4x \, \text{Tr} \Big\{ \frac{1}{2g_a^2} F_{\mu\nu}^2 + 2 \, K_\Phi \, D_\mu \bar{\Phi} D^\mu \Phi \Big\} ,
\tag{2.19}
$$

where we have introduced the Kähler metric for Φ, namely

$$
K_\Phi = \frac{1}{t_2^{(3)} u_2^{(3)}} .
\tag{2.20}
$$

---

[8] We normalize the generators T_A of the gauge group such that $\text{Tr}\,(T_A T_B) = \frac{1}{2}\delta_{AB}$.

This Kähler metric can be obtained directly also from a 3-point scattering amplitude involving one of the (closed string) geometric moduli and two scalar fields, as explained for example in Refs. [56, 53], after appropriate changes from the string to the supergravity basis.

Comparing (2.19) with the bosonic kinetic terms in (2.13), we see that the relation between the canonically normalized field ϕ appearing in the string vertex operators and the field Φ in the supergravity basis is

$$\phi = g_a \sqrt{K_\Phi}\, \Phi \ . \tag{2.21}$$

2.2 The matter sector

We now want to add $\mathcal{N} = 2$ hyper-multiplets in this orbifold set up. The simplest possibility to do this is to add a second stack of fractional D9 branes (flavor branes) which carry a different representation of the orbifold group as compared to the color branes considered so far. The massless open strings stretching between the flavor branes and the color branes account precisely for $\mathcal{N} = 2$ hyper-multiplets in the fundamental representation of the gauge group $\mathrm{SU}(N_a)$. However, we can be more general than this and introduce *magnetized* flavor D9 branes. To distinguish them from the color branes, we will denote their various parameters with a subscript b. For example, N_b will be their number and $n_b^{(i)}$ will be their wrapping number around the i-th torus.

Introducing a magnetic flux on the i-th torus for the flavor branes amounts to pick a U(1) subgroup in the Cartan subalgebra of $\mathrm{U}(n_b^{(i)})$ and turn on a constant magnetic field[9] $F_b^{(i)}$, namely

$$F_b^{(i)} = f_b^{(i)} \, dX^{2i+2} \wedge dX^{2i+3} = \mathrm{i}\, \frac{f_b^{(i)}}{T_2^{(i)}}\, dZ^i \wedge d\bar{Z}^i = \frac{f_b^{(i)}}{\sqrt{G^{(i)}}}\, J^{(i)} \ , \tag{2.22}$$

where in the last step we have introduced the Kähler form $J^{(i)}$. The generalized Dirac quantization condition requires that the first Chern class $c_1(F_b^{(i)})$ be an integer, namely

$$c_1(F_b^{(i)}) = \frac{1}{2\pi} \int_{\mathcal{T}_2^{(i)}} \mathrm{Tr}(F_b^{(i)}) = \frac{1}{2\pi}(2\pi\sqrt{\alpha'})^2 n_b^{(i)} f_b^{(i)} = m_b^{(i)} \in \mathbb{Z} \ , \tag{2.23}$$

that is

$$2\pi\alpha' f_b^{(i)} = \frac{m_b^{(i)}}{n_b^{(i)}} \ . \tag{2.24}$$

The total magnetic field is then $F_b = F_b^{(1)} + F_b^{(2)} + F_b^{(3)}$. In order to preserve at least $\mathcal{N} = 1$ supersymmetry in the bulk, the magnetic field has to satisfy the relation

$$J \wedge J \wedge \hat{F}_b = \frac{1}{3}\, \hat{F}_b \wedge \hat{F}_b \wedge \hat{F}_b \ , \tag{2.25}$$

---

[9]Even if more general magnetizations could be introduced, for simplicity we will consider only "diagonal" magnetic fields which respect the factorized structure of the internal toroidal space.

where $\hat{F}_b = 2\pi\alpha' F_b$ and J is the total Kähler form $J = \sum_i J^{(i)}$. Setting

$$2\pi\alpha' \frac{f_b^{(i)}}{T_2^{(i)}} = \tan \pi\nu_b^{(i)} \quad \text{with} \quad 0 \leq \nu_b^{(i)} < 1 \;, \tag{2.26}$$

it is easy to see that the supersymmetry requirement (2.25) is fulfilled if [10]

$$\nu_b^{(1)} - \nu_b^{(2)} - \nu_b^{(3)} = 0 \;. \tag{2.27}$$

If we want to have the same $\mathcal{N} = 2$ supersymmetry which is realized by the orbifold (2.7), we have to set

$$\nu_b^{(3)} = 0 \quad \text{and hence} \quad \nu_b^{(1)} = \nu_b^{(2)} \;. \tag{2.28}$$

This implies that the open strings stretching between the flavor branes and the color branes (*i.e.* the D9$_b$/D9$_a$ strings) are twisted only along the directions of the first two tori. More specifically, the internal string coordinates Z^i and Ψ^i defined in (2.3) satisfy, for $i = 1, 2$, the following twisted monodromy properties

$$Z^i\big(e^{2\pi i}z\big) = e^{2\pi i\nu_b^{(i)}} Z^i(z) \quad \text{and} \quad \Psi^i\big(e^{2\pi i}z\big) = \eta\, e^{2\pi i\nu_b^{(i)}} \Psi^i(z) \;, \tag{2.29}$$

where $\eta = +1$ for the NS sector and $\eta = -1$ for the R sector. On the other hand, Z^3 and Ψ^3 have the usual untwisted properties.

Let us now describe the physical massless states of the D9$_b$/D9$_a$ strings, starting from the NS sector. To write the vertex operators it is convenient to introduce the following notation

$$\sigma(z) \equiv \prod_{i=1}^{2} \sigma_{\nu_b^{(i)}}(z) \;, \quad s(z) \equiv \prod_{i=1}^{2} S_{\nu_b^{(i)}} \;, \tag{2.30}$$

where $\sigma_{\nu_b^{(i)}}$ and $S_{\nu_b^{(i)}}$ are respectively the bosonic and fermionic twist fields in the i-th torus whose conformal dimensions are

$$h_\sigma^{(i)} = \frac{1}{2}\nu_b^{(i)}\big(1 - \nu_b^{(i)}\big) \quad \text{and} \quad h_S^{(i)} = \frac{1}{2}\big(\nu_b^{(i)}\big)^2 \;. \tag{2.31}$$

Then, the physical massless states are described by the following vertex operators:

$$\begin{aligned}
V_q(z) &= (2\pi\alpha')^{\frac{1}{2}}\, q\; \sigma(z) :\bar{\Psi}^1(z)s(z): e^{-\varphi(z)}\, e^{ip_\mu X^\mu(z)} \;, \\
V_{\tilde{q}^\dagger}(z) &= (2\pi\alpha')^{\frac{1}{2}}\, \tilde{q}^\dagger\; \sigma(z) :\bar{\Psi}^2(z)s(z): e^{-\varphi(z)}\, e^{ip_\mu X^\mu(z)} \;,
\end{aligned} \tag{2.32}$$

which can be easily checked to have conformal dimension 1 for $p^2 = 0$ if $\nu_b^{(1)} = \nu_b^{(2)}$.

In the R sector instead the massless states are described by the following vertex operators

$$\begin{aligned}
V_\chi(z) &= (2\pi\alpha')^{\frac{3}{4}}\, \chi^\alpha\, S_\alpha(z)\, \sigma(z)\Sigma(z)S_-(z)\, e^{-\frac{1}{2}\varphi(z)}\, e^{ip_\mu X^\mu(z)} \;, \\
V_{\tilde{\chi}^\dagger}(z) &= (2\pi\alpha')^{\frac{3}{4}}\, \tilde{\chi}^\dagger_{\dot{\alpha}}\, S^{\dot{\alpha}}(z)\, \sigma(z)\Sigma(z)S_+(z)\, e^{-\frac{1}{2}\varphi(z)}\, e^{ip_\mu X^\mu(z)} \;,
\end{aligned} \tag{2.33}$$

---

[10]Other solutions of (2.25) are $-\nu_b^{(1)} - \nu_b^{(2)} + \nu_b^{(3)} = 0$; $-\nu_b^{(1)} + \nu_b^{(2)} - \nu_b^{(3)} = 0$; $\nu_b^{(1)} + \nu_b^{(2)} + \nu_b^{(3)} = 2$. They are all related to the solution (2.27) by obvious changes.

where

$$\Sigma(z) = \prod_{i=1}^{2} S_{\nu_b^{(i)} - \frac{1}{2}}(z) \tag{2.34}$$

and $S_\pm$ are the spin fields in the untwisted directions of the third torus. Again, one can easily check that these vertex operators have conformal dimension 1 for $p^2 = 0$.

In all the above vertices the polarizations, which carry a color index in the fundamental representation of $\mathrm{SU}(N_a)$, have canonical dimensions and are odd under $\mathbb{Z}_2$, since they describe open strings that connect fractional branes belonging to different irreducible representations of the orbifold group. Consequently we must require that the operator part in (2.32) and (2.33) be also odd under $\mathbb{Z}_2$ so that altogether the complete vertices can survive the orbifold projection. In particular this implies that the twisted part of the R ground states, described by $\sigma(z)\Sigma(z)$, must be declared odd under $\mathbb{Z}_2$ while the twisted part of the NS ground states, described by $\sigma(z)s(z)$, must be declared even. The vertices (2.32) and (2.33) are connected to each other by the same eight supercharges (2.12) that are selected by the $\mathbb{Z}_2$ orbifold, and thus their polarizations form a hyper-multiplet representation of $\mathcal{N} = 2$ supersymmetry. More precisely, taking into account the multiplicity of the (a, b) intersection, they can be organized into N_F hyper-multiplets whose components in the following will be denoted as $\left(q_f, \tilde{q}_f^\dagger, \chi_f, \tilde{\chi}_f^\dagger\right)$ with $f = 1, \dots, N_F$.

The $\mathrm{D}9_a/\mathrm{D}9_b$ strings with opposite orientation have a completely similar structure; at the massless level the physical vertex operators are

$$\begin{aligned}
V_{q^\dagger}(z) &= (2\pi\alpha')^{\frac{1}{2}} q^\dagger \; \bar{\sigma}(z) : \Psi^1(z)\bar{s}(z) : \mathrm{e}^{-\varphi(z)} \, \mathrm{e}^{\mathrm{i}p_\mu X^\mu(z)} \;, \\
V_{\tilde{q}}(z) &= (2\pi\alpha')^{\frac{1}{2}} \tilde{q} \; \bar{\sigma}(z) : \Psi^2(z)\bar{s}(z) : \mathrm{e}^{-\varphi(z)} \, \mathrm{e}^{\mathrm{i}p_\mu X^\mu(z)}
\end{aligned} \tag{2.35}$$

in the NS sector, and

$$\begin{aligned}
V_{\chi^\dagger}(z) &= (2\pi\alpha')^{\frac{3}{4}} \chi_{\dot{\alpha}}^\dagger \; S^{\dot{\alpha}}(z) \, \bar{\sigma}(z)\bar{\Sigma}(z)S_+(z) \, \mathrm{e}^{-\frac{1}{2}\varphi(z)} \, \mathrm{e}^{\mathrm{i}p_\mu X^\mu(z)} \;, \\
V_{\tilde{\chi}}(z) &= (2\pi\alpha')^{\frac{3}{4}} \tilde{\chi}^\alpha \; S_\alpha(z) \, \bar{\sigma}(z)\bar{\Sigma}(z)S_-(z) \, \mathrm{e}^{-\frac{1}{2}\varphi(z)} \, \mathrm{e}^{\mathrm{i}p_\mu X^\mu(z)}
\end{aligned} \tag{2.36}$$

in the R sector. Here we have defined the anti-twist fields as follows:

$$\bar{\sigma}(z) \equiv \prod_{i=1}^{2} \sigma_{1-\nu_b^{(i)}}(z) \;, \quad \bar{s}(z) \equiv \prod_{i=1}^{2} S_{-\nu_b^{(i)}}(z) \;, \quad \bar{\Sigma}(z) \equiv \prod_{i=1}^{2} S_{\frac{1}{2}-\nu_b^{(i)}}(z) \;. \tag{2.37}$$

The vertices (2.35) and (2.36) are conjugate to the ones in (2.32) and (2.33) respectively.

By computing all tree-level scattering amplitudes among the above vertex operators and those of gauge sector, and taking the field theory limit $\alpha' \to 0$, one can obtain the $\mathcal{N} = 2$ action for hyper-multiplets coupled to a vector multiplet. For example, from the computation of a 3-point function between a gluon, a scalar of the hyper-multiplet and its conjugate, one can reconstruct the kinetic terms

$$\int d^4x \; \sum_{f=1}^{N_F} \left\{ D_\mu q^{\dagger f} \, D^\mu q_f + D_\mu \tilde{q}^f \, D^\mu \tilde{q}_f^\dagger \right\} \;, \tag{2.38}$$

where we have explicitly indicated the sum over the flavor indices and suppressed the color indices. Similarly, from other 3-point functions one can obtain the various Yukawa interactions, like for example

$$\int d^4x \sum_{f=1}^{N_F} \tilde{\chi}^f \, \phi \, \chi_f \; . \tag{2.39}$$

In the supergravity basis it is customary to use fields with a different normalization and write for example the kinetic term for the scalars of the hyper-multiplet as

$$\int d^4x \; \sum_{f=1}^{N_F} K_Q \Big\{ D_\mu Q^{\dagger f} \, D^\mu Q_f + D_\mu \tilde{Q}^f \, D^\mu \tilde{Q}^\dagger_f \Big\} \; . \tag{2.40}$$

Upon comparison with (2.38), we see that the relation between the canonically normalized fields q and $\tilde{q}$ appearing in the string vertex operators and the fields Q and $\tilde{Q}$ of the supergravity basis is

$$q = \sqrt{K_Q}\, Q \quad \text{and} \quad \tilde{q} = \sqrt{K_Q}\, \tilde{Q} \; . \tag{2.41}$$

On the other hand, using a $\mathcal{N}=1$ language in the supergravity basis, the various Yukawa couplings can be encoded in the holomorphic superpotential

$$W = \sum_{f=1}^{N_F} \tilde{Q}^f \, \Phi \, Q_f \; , \tag{2.42}$$

where we have adopted for the chiral superfields the same notation used for their bosonic components.

By explicitly writing the relation between the Yukawa couplings in the canonical basis (see *e.g.* (2.39)) and those in the supergravity basis derived from the $\mathcal{N}=1$ superpotential (2.42), we obtain

$$1 = \mathrm{e}^{K/2} \left(\sqrt{K_Q}\right)^{-2} \left(g_a \sqrt{K_\Phi}\right)^{-1} \; , \tag{2.43}$$

where the factor $\mathrm{e}^{K/2}$ is the contribution of the bulk supergravity Kähler potential. Clearly, we can rewrite (2.43) also as

$$\mathrm{e}^{K/2} \, K_Q^{-1} = g_a \sqrt{K_\Phi} \; , \tag{2.44}$$

which will be useful later [11]. Using (2.14), the expression for the Kähler potential K given in (2.6) and the Kähler metric K_Φ given in (2.20), we deduce that

$$K_Q = \frac{1}{\left(t_2^{(1)} t_2^{(2)} u_2^{(1)} u_2^{(2)}\right)^{1/2}} \; . \tag{2.45}$$

This expression for K_Q agrees with the one mentioned in Ref. [33]. It is worth pointing out that also the metric (2.45) can be reconstructed from a 3-point scattering amplitude along the lines discussed in Refs. [56, 53], after the appropriate changes between the string and the supergravity basis are taken into account.

---

[11]In Section 6 we will rewrite this relation in a full fledged $\mathcal{N}=2$ notation (see Eq. (6.7)).

In the following we will consider also the case in which the hyper-multiplets are massive with a $\mathcal{N} = 2$ invariant mass term given by

$$W_M = \sum_{f=1}^{N_F} M_f \, \tilde{Q}^f \, Q_f \, . \tag{2.46}$$

From this superpotential we immediately see that the corresponding mass parameters m_f appearing in the canonically normalized action of the string basis are

$$m_f = e^{K/2} \, K_Q^{-1} \, M_f = g_a \sqrt{K_\Phi} \, M_f \, , \tag{2.47}$$

where in the last step we have used (2.44). Comparing this expression with (2.21), we see that m_f and ϕ are related to the corresponding quantities M_f and Φ in the supergravity basis in the same way.

2.3 Generalizations

The above construction can be easily generalized in several ways. For example, if a background B field is turned on in the internal space, see (2.1), the magnetic flux $2\pi\alpha' f_b^{(i)}$ gets replaced by $\hat{f}_b^{(i)} \equiv 2\pi\alpha' f_b^{(i)} - T_1^{(i)}$, so that (2.26) becomes

$$\tan \pi \nu_b^{(i)} = \frac{m_b^{(i)} - n_b^{(i)} T_1^{(i)}}{n_b^{(i)} T_2^{(i)}} \, , \tag{2.48}$$

where the quantization condition (2.24) has been taken into account. Note that in the presence of B also the color branes acquire intrinsic twist parameters given by

$$\tan \pi \nu_a^{(i)} = -\frac{T_1^{(i)}}{T_2^{(i)}} \tag{2.49}$$

and the monodromy properties of the D9$_b$/D9$_a$ strings depend on the relative twist parameters

$$\nu_{ba}^{(i)} = \nu_b^{(i)} - \nu_a^{(i)} \tag{2.50}$$

which must replace $\nu_b^{(i)}$ in the various vertex operators like (2.32) and (2.33). We can further generalize this by wrapping the color branes $n_a^{(i)}$ times on the i-th torus and turning on a magnetic field on their world volume with integer magnetic numbers $m_a^{(i)}$. In this way the intrinsic twist parameters $\nu_a^{(i)}$ of the color branes have the same expression as (2.48) with the subscript b replaced by a.

Non-trivial wrapping and magnetic numbers for the color branes also influence the explicit expressions of the various quantities in the effective gauge theory. For example, the gauge coupling constant g_a turns out to be given by

$$\frac{1}{g_a^2} = \frac{1}{4\pi} \, e^{-\phi_{10}} \prod_{i=1}^{3} \left| n_a^{(i)} T^{(i)} - m_a^{(i)} \right| = s_2 \left| \ell_a^{(1)} \ell_a^{(2)} \ell_a^{(3)} \right| \, , \tag{2.51}$$

where we have defined

$$\ell_a^{(i)} = \frac{n_a^{(i)} T^{(i)} - m_a^{(i)}}{T_2^{(i)}} \ . \tag{2.52}$$

Note that if we use the supersymmetry relation

$$\prod_i \frac{\widehat{f}_a^{(i)}}{T_2^{(i)}} = \sum_i \frac{\widehat{f}_a^{(i)}}{T_2^{(i)}} \tag{2.53}$$

for the quantities $\widehat{f}_a^{(i)} \equiv 2\pi\alpha' f_a^{(i)} - T_1^{(i)}$, which follows from the obvious extension of (2.25), we can rewrite (2.51) as

$$\frac{1}{g_a^2} = n_a^{(1)} n_a^{(2)} n_a^{(3)} \left| s_2 - \frac{1}{4\pi} \left(\widehat{f}_a^{(1)} \widehat{f}_a^{(2)} t_2^{(3)} + \widehat{f}_a^{(2)} \widehat{f}_a^{(3)} t_2^{(1)} + \widehat{f}_a^{(3)} \widehat{f}_a^{(1)} t_2^{(2)} \right) \right| \ . \tag{2.54}$$

By repeating the same analysis of the previous subsections when the color branes are magnetized, one finds that the Kähler metric for the adjoint scalar field Φ is

$$K_\Phi = \frac{1}{t_2^{(3)} u_2^{(3)}} \left| \frac{\ell_a^{(1)} \ell_a^{(2)}}{\ell_a^{(3)}} \right| \tag{2.55}$$

and that the Kähler metric for the fundamental chiral multiplets Q and $\tilde{Q}$ is

$$K_Q = \frac{1}{\left(t_2^{(1)} t_2^{(2)} u_2^{(1)} u_2^{(2)} \right)^{1/2}} \left| \ell_a^{(3)} \right| \ . \tag{2.56}$$

These expressions reduce to those given respectively in (2.20) and (2.45) when the color branes are not magnetized and are trivially wrapped on the internal space, since in this case $|\ell_a^{(i)}| \to 1$ for all i.

Performing a T-duality transformation

$$T^{(i)} \to -\frac{1}{U^{(i)}} \quad , \quad U^{(i)} \to -\frac{1}{T^{(i)}} \tag{2.57}$$

with the four-dimensional dilaton ϕ_4 kept fixed, we can translate our results for magnetized D9 branes into those for intersecting D6 branes of the type IIA theory. Under this transformation, $t_2^{(i)}$ and $u_2^{(i)}$ are interchanged, while

$$\ell_a^{(i)} \to -\frac{\bar{U}^{(i)}}{U_2^{(i)}} \left(n_a^{(i)} + U^{(i)} m_a^{(i)} \right) \ . \tag{2.58}$$

We can therefore see that, after T-duality, the Kähler metrics (2.55) and (2.56) are a generalization of those presented in Ref. [33] for intersecting D branes on rectangular tori (*i.e.* $U_1^{(i)} = 0$). Notice also that when all branes are magnetized, the number N_F of fundamental hyper-multiplets associated to the strings stretching between the D9$_b$ and the D9$_a$ branes is given by

$$N_F = N_b \, I_{ba} = N_b \, I_{ab} \ , \tag{2.59}$$

where

$$I_{ab} = \prod_{i=1}^{2} \left(m_a^{(i)} n_b^{(i)} - m_b^{(i)} n_a^{(i)} \right) = I_{ba} \tag{2.60}$$

represents the number of Landau levels for the (a, b) intersection.

We finally observe that in a generic toroidal orbifold compactification with wrapped branes there are unphysical closed string tadpoles that must be canceled to have a globally consistent model. Usually this cancellation is achieved by introducing an orientifold projection and suitable orientifold planes. Like in other cases treated in the literature, in this paper we take a "local" point of view focusing only on some intersections and assume that the model can be made fully consistent with the orientifold projection.

3. $\mathcal{N} = 2$ instanton calculus from the string perspective

We now consider instanton effects in the $\mathcal{N} = 2$ gauge theories presented in the previous section. In this stringy set-up instanton contributions can be obtained by adding fractional Euclidean D5 branes (nowadays called E5 branes) that completely wrap the internal manifold $\frac{\mathcal{T}_2^{(1)} \times \mathcal{T}_2^{(2)}}{\mathbb{Z}_2} \times \mathcal{T}_2^{(3)}$, and hence describe point-like configurations from the four-dimensional point of view. In general these E5 branes can be chosen with a representation of the $\mathbb{Z}_2$ orbifold group on the Chan-Paton factors and/or with magnetic fluxes that are different from the ones of the color $D9_a$ branes. If that is the case, then the E5 branes represent "exotic" instantons whose properties are different from those of the ordinary gauge theory instantons. Recently, these "exotic" configurations have been the subject of active investigations [24]–[36] from several different points of view. Here we start by considering E5 branes that have the same characteristics of the color $D9_a$ branes, except for their dimensions. Therefore, we call them $E5_a$ branes. As we will see in detail later, these $E5_a$ branes represent ordinary gauge instantons for the SYM theory on the $D9_a$ branes. However, they are "exotic" instantons with respect to the gauge theory defined on the flavor $D9_b$, and thus our results can be useful also for the new developments.

The addition of k $E5_a$ branes introduces new types of excitations associated to open strings with at least one end-point on the instantonic branes, namely the $E5_a/E5_a$ strings, the $D9_a/E5_a$ (or $E5_a/D9_a$) strings and the $D9_b/E5_a$ (or $E5_a/D9_b$) strings. In all these instantonic sectors, due to the Dirichlet-Dirichlet or mixed Dirichlet-Neumann boundary conditions in the four non-compact directions, the open string excitations do not carry any momentum and hence represent moduli rather than dynamical fields in space-time. They however can carry (discretized) momentum along the compact directions. Therefore we can distinguish the open string states into those which do not carry any momentum in any directions and those which do. The lightest excitations of the first type are truly instanton moduli while those of the second type represent genuine string corrections whose relevance for the effective theory will be elucidated in the following.

3.1 Instanton moduli

We now briefly list the instanton moduli for our $\mathcal{N} = 2$ model which we distinguish into neutral, charged and flavored ones.

The neutral instanton sector The neutral instanton sector comprises the zero-modes of open strings with both ends on the $E5_a$ branes. These modes are usually referred to as neutral because they do not transform under the gauge group. In the NS sector, after the $\mathbb{Z}_2$ orbifold projection, we find six physical bosonic excitations that can be conveniently organized in a vector a_μ and a complex scalar χ, and also three auxiliary excitations D_c ($c = 1, 2, 3$). The corresponding vertex operators are

$$V_a(z) = g_{5_a} (2\pi\alpha')^{\frac{1}{2}} a_\mu \psi^\mu(z) \, e^{-\varphi(z)} \, , \tag{3.1a}$$

$$V_\chi(z) = \chi (\pi\alpha')^{\frac{1}{2}} \Psi^3(z) \, e^{-\varphi(z)} \, , \tag{3.1b}$$

$$V_D(z) = D_c (\pi\alpha') \, \bar{\eta}^c_{\mu\nu} \, \psi^\nu(z) \psi^\mu(z) \, , \tag{3.1c}$$

where $\bar{\eta}^c_{\mu\nu}$ are the three anti-self-dual 't Hooft symbols and g_{5_a} is the (dimensionful) coupling constant on the $E5_a$, namely

$$g_{5_a} = \frac{g_a}{4\pi^2\alpha'} \tag{3.2}$$

with g_a given in (2.51). In the R sector, after the orbifold projection (2.11), we find four chiral fermionic zero-modes $M^{\alpha A}$ described by the vertex operators

$$V_{M^1}(z) = \frac{g_{5_a}}{\sqrt{2}} (2\pi\alpha')^{\frac{3}{4}} M^{\alpha 1} S_\alpha(z) S_{+-+}(z) \, e^{-\frac{1}{2}\varphi(z)} \, ,$$
$$V_{M^2}(z) = \frac{g_{5_a}}{\sqrt{2}} (2\pi\alpha')^{\frac{3}{4}} M^{\alpha 2} S_\alpha(z) S_{-++}(z) \, e^{-\frac{1}{2}\varphi(z)} \, , \tag{3.3}$$

and four anti-chiral zero-modes $\lambda_{\dot{\alpha}A}$, described by the vertices

$$V_{\lambda_1}(z) = \lambda_{\dot{\alpha}1} (2\pi\alpha')^{\frac{3}{4}} S^{\dot{\alpha}}(z) S^{+--}(z) \, e^{-\frac{1}{2}\varphi(z)} \, ,$$
$$V_{\lambda_2}(z) = \lambda_{\dot{\alpha}2} (2\pi\alpha')^{\frac{3}{4}} S^{\dot{\alpha}}(z) S^{-+-}(z) \, e^{-\frac{1}{2}\varphi(z)} \, . \tag{3.4}$$

All polarizations in the vertex operators (3.1), (3.3) and (3.4) are $k \times k$ matrices and transform in the adjoint representation of $U(k)$. It is worth noticing that if the Yang-Mills coupling constant g_a is kept fixed when $\alpha' \to 0$, then the dimensionful coupling g_{5_a} in (3.2) blows up. Thus, some of the vertex operators have been rescaled with factors of g_{5_a} (like in (3.1a) and (3.3)) in order to yield non-trivial interactions when $\alpha' \to 0$ [18]. As a consequence of this rescaling some of the moduli acquire unconventional scaling dimensions which, however, are the right ones for their interpretation as parameters of an instanton solution [16, 18]. For instance, the a_μ's have dimensions of (length) and are related to the positions of the (multi-)centers of the instanton, while $M^{\alpha A}$ have dimensions of (length)$^{\frac{1}{2}}$ and are the fermionic partners of the instanton centers. Furthermore, if we write the $k \times k$ matrices a^μ and $M^{\alpha A}$ as

$$a^\mu = x_0^\mu \, \mathbb{1}_{k \times k} + y_c^\mu \, T^c \quad , \quad M^{\alpha A} = \theta^{\alpha A} \, \mathbb{1}_{k \times k} + \zeta_c^{\alpha A} \, T^c \, , \tag{3.5}$$

where T^c are the generators of $SU(k)$, then the instanton center of mass, x_0^μ, and its fermionic partners, $\theta^{\alpha A}$, can be identified respectively with the bosonic and fermionic coordinates of the $\mathcal{N} = 2$ superspace.

15

The charged instanton sector The charged instanton sector contains the zero-modes of the open strings stretching between the color $D9_a$ branes and the $E5_a$ branes, which transform in the fundamental representation of the gauge group. In the NS sector there are two physical bosonic moduli $w_{\dot\alpha}$ with dimension of (length) whose vertex operator is

$$V_w(z) = \frac{g_{5a}}{\sqrt{2}} \, (2\pi\alpha')^{\frac{1}{2}} \, w_{\dot\alpha} \, \Delta(z) S^{\dot\alpha}(z) \, \mathrm{e}^{-\varphi(z)} \; . \tag{3.6}$$

Here Δ is the twist operator with conformal weight $1/4$ which changes the boundary conditions of the uncompact coordinates X^μ from Neumann to Dirichlet. In the R sector there are two fermionic moduli μ^A with dimension of $(\mathrm{length})^{1/2}$ whose vertices are

$$\begin{aligned}
V_{\mu^1}(z) &= \frac{g_{5a}}{\sqrt{2}} \, (2\pi\alpha')^{\frac{3}{4}} \, \mu^1 \, \Delta(z) S_{+-+}(z) \, \mathrm{e}^{-\frac{1}{2}\varphi(z)} \; , \\
V_{\mu^2}(z) &= \frac{g_{5a}}{\sqrt{2}} \, (2\pi\alpha')^{\frac{3}{4}} \, \mu^2 \, \Delta(z) S_{-++}(z) \, \mathrm{e}^{-\frac{1}{2}\varphi(z)} \; .
\end{aligned} \tag{3.7}$$

Both in (3.6) and (3.7) the polarizations are $N_a \times k$ matrices which transform in the bi-fundamental representation $(N_a, \bar{k})$ of $U(N_a) \times U(k)$. Notice that these vertex operators are even under the $\mathbb{Z}_2$ orbifold projection (2.11). The charged moduli associated to the open strings stretching from the $E5_a$ branes to the $D9_a$'s, denoted by $\bar{w}_{\dot\alpha}$ and $\bar\mu^A$, transform in the $(\bar{N}_a, k)$ representation and are described by vertex operators of the same form as (3.6) and (3.7) except for the replacement of $\Delta(z)$ by the anti-twist $\bar\Delta(z)$, corresponding to mixed Dirichlet-Neumann boundary conditions along the four space-time directions. It is worth pointing out that $\bar\mu^A$ are *not* the conjugates of μ^A. This fact has important consequences for our purposes, as we will discuss in Sect. 4.

The flavored instanton sector The flavored instanton sector corresponds to the open strings that stretch between the flavor $D9_b$ branes and the $E5_a$ branes. In this case the four non-compact directions have mixed Neumann-Dirichlet boundary conditions while the complex coordinates along the first two tori are twisted with parameters $\nu_{ba}^{(1)} = \nu_{ba}^{(2)}$ due to the different magnetic fluxes at two end-points. As a consequence of this, there are no bosonic physical zero-modes in the NS sector and the only physical excitations are fermionic moduli with dimension of $(\mathrm{length})^{\frac{1}{2}}$ from the R sector, whose vertices are given by

$$V_{\mu'}(z) = \frac{g_{5a}}{\sqrt{2}} \, (2\pi\alpha')^{\frac{3}{4}} \, \mu' \, \Delta(z) \, \sigma(z)\Sigma(z) \, S_-(z) \, \mathrm{e}^{-\frac{1}{2}\varphi(z)} \; . \tag{3.8}$$

Notice that this vertex operator is even under the $\mathbb{Z}_2$ orbifold group, since both the operator part and the polarization are odd under $\mathbb{Z}_2$, in complete analogy to what happens to the fermionic vertices (2.36) of the flavored matter. Finally, we recall that the zero-modes for the $E5_a/D9_b$ open strings with opposite orientation are described by the vertex operators

$$V_{\bar\mu'}(z) = \frac{g_{5a}}{\sqrt{2}} \, (2\pi\alpha')^{\frac{3}{4}} \, \bar\mu' \, \bar\Delta(z) \, \bar\sigma(z)\bar\Sigma(z) \, S_-(z) \, \mathrm{e}^{-\frac{1}{2}\varphi(z)} \; . \tag{3.9}$$

Taking into account the multiplicity of the (a, b) intersection, we will have altogether N_F fermionic moduli of each type which will be denoted as μ'_f and $\bar\mu'^f$ with $f = 1, \ldots, N_F$.

The physical moduli we have listed above, collectively called $\mathcal{M}_k$, are in one-to-one correspondence with the ADHM moduli of $\mathcal{N} = 2$ gauge instantons (for a more detailed discussion see, for instance, Ref. [16] and references therein). In all instantonic sectors we can construct many other open string states that carry a discretized momentum along the compact directions and/or have some bosonic or fermionic string oscillators. All these "massive" states, however, are not physical, *i.e.* they cannot be described by vertex operators of conformal dimension one, but, as we will see later, they can play a role as internal states circulating in open string loop diagrams.

3.2 Instanton partition function

Having identified the ADHM moduli, in analogy with the instanton calculus in field theory we define the k-instanton partition function as the "functional" integral over the instanton moduli, namely

$$Z_k = \mathcal{C}_k \int d\mathcal{M}_k \; \mathrm{e}^{-S(\mathcal{M}_k)} \; , \tag{3.10}$$

where $\mathcal{C}_k$ is a dimensional normalization factor which compensates for the dimensions of the integration measure $d\mathcal{M}_k$, and $S(\mathcal{M}_k)$ is the moduli effective action which accounts for all possible interactions among the instanton moduli in the limit $\alpha' \to 0$ (with g_a fixed) at any order of string perturbation theory. This action can be obtained by computing the field theory limit of all scattering amplitudes with the vertex operators of the ADHM moduli inserted on boundaries of open string world-sheets of any topology. Formally we can write

$$\begin{aligned}
-S(\mathcal{M}_k) &= \sum_{\text{topology}} \langle\, 1 \,\rangle_{\text{topology}} + \langle\, \mathcal{M}_k \,\rangle_{\text{topology}} \\
&= \langle\, 1 \,\rangle_{\text{disk}} + \langle\, 1 \,\rangle'_{\text{annulus}} + \cdots + \langle\, \mathcal{M}_k \,\rangle_{\text{disk}} + \langle\, \mathcal{M}_k \,\rangle'_{\text{annulus}} + \cdots \; ,
\end{aligned} \tag{3.11}$$

where $\langle\, 1 \,\rangle_{\text{topology}}$ denotes the vacuum amplitudes and $\langle\, \mathcal{M}_k \,\rangle_{\text{topology}}$ the amplitudes with moduli insertions. Since the functional integration over the ADHM moduli $\mathcal{M}_k$ is explicitly performed in (3.10), to avoid double counting only the contribution of the "massive" string excitations has to be taken into account in computing the higher order terms of $S(\mathcal{M}_k)$. This is the reason of the $'$ notation in the annulus contributions, which reminds that only the "massive" instantonic string excitations must circulate in the loop.

In the semi-classical approximation, which is typical of the instanton calculus, it is enough to consider the vacuum amplitudes up to one loop and the moduli interactions at tree level since, as we will see momentarily,

$$\langle\, 1 \,\rangle_{\text{disk}} = \mathcal{O}\big(g_a^{-2}\big) \quad , \quad \langle\, 1 \,\rangle'_{\text{annulus}} = \mathcal{O}\big(g_a^0\big) \quad , \quad \langle\, \mathcal{M}_k \,\rangle_{\text{disk}} = \mathcal{O}\big(g_a^0\big) \; , \tag{3.12}$$

while $\langle\, \mathcal{M}_k \,\rangle_{\text{annulus}}$ or the higher topology contributions are of higher order in the Yang-Mills coupling constant. Thus, in this approximation the k-instanton partition function is

$$Z_k = \mathcal{C}_k \; \mathrm{e}^{\langle\, 1 \,\rangle_{\text{disk}} + \langle\, 1 \,\rangle'_{\text{annulus}}} \int d\mathcal{M}_k \; \mathrm{e}^{\langle\, \mathcal{M}_k \,\rangle_{\text{disk}}} \; . \tag{3.13}$$

Let us now discuss the various terms of this expression in turn.

The dimensional factor $\mathcal{C}_k$ can be easily determined by counting the dimensions (measured in units of α') of the various moduli $\mathcal{M}_k$ as given in the previous subsections, and the result is

$$\mathcal{C}_k = \left(\sqrt{\alpha'}\right)^{-(2N_a - N_F)k} . \tag{3.14}$$

Notice the appearance of the 1-loop coefficient $b_1 = (2N_a - N_F)$ of the β-function of the $\mathcal{N} = 2$ SYM theory[12].

The vacuum amplitude at tree level $\langle 1 \rangle_{\mathrm{disk}}$ is nothing but the topological normalization of the a disk whose boundary lies on the k E5$_a$ branes, which is [1, 18]

$$\langle 1 \rangle_{\mathrm{disk}} \equiv \mathcal{D}_{5_a} = -\frac{8\pi^2}{g_a^2} k , \tag{3.15}$$

where g_a, given in (2.51), is interpreted as the Yang-Mills coupling constant at the string scale $\sqrt{\alpha'}$. Notice that the vacuum amplitude (3.15) is also minus the value of the classical instanton action. Using these results we have

$$\mathcal{C}_k \, e^{\langle 1 \rangle_{\mathrm{disk}}} = \Lambda^{(2N_a - N_F)k} , \tag{3.16}$$

where Λ is the renormalization group invariant scale of the $\mathcal{N} = 2$ gauge theory on the color branes. On the other hand these factors do not seem to have an obvious interpretation in terms of the four-dimensional field theory living on the flavor branes for which the E5$_a$ branes would represent "exotic" instantons of truly stringy nature.

The 1-loop vacuum amplitude

$$\langle 1 \rangle_{\mathrm{annulus}} \equiv \mathcal{A}_{5_a} \tag{3.17}$$

is also contributing to the overall normalization factor of the partition function through its "primed" part. We will give its explicit expression in the next section, where we will also discuss its meaning and relevance for the instanton calculus.

The last object appearing in Z_k is the tree-level moduli interaction term $\langle \mathcal{M}_k \rangle_{\mathrm{disk}}$ which can be computed following the procedure explained in Ref. [18] from the disk scattering amplitudes among all ADHM moduli in the limit $\alpha' \to 0$ (with g_a fixed). The result is [16, 22]

$$
\begin{aligned}
\langle \mathcal{M}_k \rangle_{\mathrm{disk}} = \; & \mathrm{tr}_k \Big\{ 2 \left[\chi^\dagger, a_\mu \right] \left[\chi, a^\mu \right] - \chi^\dagger \bar{w}_{\dot\alpha} w^{\dot\alpha} \chi - \chi \bar{w}_{\dot\alpha} w^{\dot\alpha} \chi^\dagger \\
& - \mathrm{i} \frac{\sqrt{2}}{2} \bar{\mu}^A \epsilon_{AB} \mu^B \chi^\dagger + \mathrm{i} \frac{\sqrt{2}}{4} M^{\alpha A} \epsilon_{AB} \left[\chi^\dagger, M_\alpha^B \right] - \mathrm{i} \frac{\sqrt{2}}{2} \sum_{f=1}^{N_F} \bar{\mu}'^f \mu'_f \chi \\
& + \mathrm{i} D_c \Big(\bar{w}_{\dot\alpha} (\tau^c)^{\dot\alpha}_{\ \dot\beta} w^{\dot\beta} + \mathrm{i} \bar{\eta}^c_{\mu\nu} \left[a^\mu, a^\nu \right] \Big) - \mathrm{i} \lambda^{\dot\alpha}_A \Big(\bar{\mu}^A w_{\dot\alpha} + \bar{w}_{\dot\alpha} \mu^A + \left[a_\mu, M^{\alpha A} \right] \sigma^\mu_{\alpha\dot\alpha} \Big) \Big\} ,
\end{aligned}
\tag{3.18}
$$

where we have explicitly indicated the sum over the flavor indices and understood the one on the color indices. Notice that the moduli D_c and $\lambda^{\dot\alpha}_A$ appear only linearly in the last two terms of (3.18) and thus act as Lagrange multipliers for the bosonic and fermionic

---

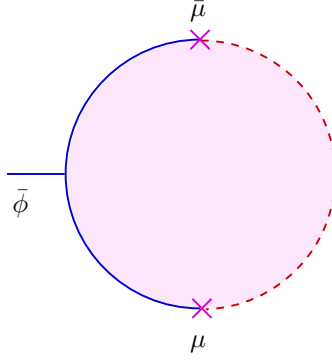[12]We define the 1-loop β-function as $\beta(g) = -(b_1/16\pi^2)g^3$.

Figure 1: The mixed disk representing the coupling of the adjoint scalar field $\bar{\phi}$ to the instanton fermionic moduli $\bar{\mu}$ and μ.

constraints of the ADHM construction. Notice also that $\langle \mathcal{M}_k \rangle_{\text{disk}}$ is indeed of $\mathcal{O}(g_a^0)$ as anticipated above, and that it does not depend on the instanton center x_0^μ nor on its super-partners $\theta^{\alpha A}$ defined in (3.5). For this reason it is convenient to separate x_0^μ and $\theta^{\alpha A}$ from the remaining centered moduli, denoted by $\widehat{\mathcal{M}}_k$, and simplify the notation by setting $\langle \mathcal{M}_k \rangle_{\text{disk}} \equiv - S_{\text{mod}}(\widehat{\mathcal{M}}_k)$. In this way we have

$$Z_k = \int d^4x_0 \, d^4\theta \, \widehat{Z}_k \; , \tag{3.19}$$

where

$$\widehat{Z}_k = \Lambda^{(2N_a - N_F)k} \, \mathrm{e}^{\mathcal{A}'_{5a}} \int d\widehat{\mathcal{M}}_k \, \mathrm{e}^{-S_{\text{mod}}(\widehat{\mathcal{M}}_k)} \tag{3.20}$$

is the centered k-instanton partition function.

3.3 Instanton induced prepotential and effective action

Let us now briefly discuss how instanton contributions to gauge field correlation functions are computed in this string set-up.

The first step is to generalize the moduli action $S_{\text{mod}}(\widehat{\mathcal{M}}_k)$ to include the interactions with gauge fields. Here for definiteness we will consider only the Coulomb branch of the $\mathcal{N} = 2$ theory, *i.e.* we will discuss the interactions with the adjoint scalar fields. In our semi-classical approximation this is achieved by computing all possible disk amplitudes with insertions of vertex operators for instanton moduli and scalar fields as well, like the one represented in Fig. 1.

The disk amplitudes that involve the adjoint scalar ϕ (or its conjugate $\bar{\phi}$) and survive in the limit $\alpha' \to 0$ give rise to the following action [16]

$$
\begin{aligned}
S_{\text{mod}}(\phi, \bar{\phi}, m; \mathcal{M}_k) = -\text{tr}_k \Big\{ & 2 \, [\chi^\dagger, a'_\mu][\chi, a'^\mu] - \big(\chi^\dagger \bar{w}_{\dot\alpha} - \bar{w}_{\dot\alpha} \, \bar{\phi}\big)\big(w^{\dot\alpha} \chi - \phi \, w^{\dot\alpha}\big) \\
& - \big(\chi \bar{w}_{\dot\alpha} - \bar{w}_{\dot\alpha} \, \phi\big)\big(w^{\dot\alpha} \chi^\dagger - \bar{\phi} \, w^{\dot\alpha}\big) - \mathrm{i}\frac{\sqrt{2}}{2} \bar{\mu}^A \epsilon_{AB}\big(\mu^B \chi^\dagger + \bar{\phi} \, \mu^B\big) \\
& + \mathrm{i}\frac{\sqrt{2}}{4} M^{\alpha A} \epsilon_{AB}[\chi^\dagger, M_\alpha^B] - \mathrm{i}\frac{\sqrt{2}}{2} \sum_{f=1}^{N_F} \bar{\mu}'^f\big(\mu'_f \chi + m_f \, \mu'\big) + S_{\text{constr}} \Big\} \; ,
\end{aligned}
\tag{3.21}
$$

where S_{constr} denotes the ADHM constraint part (*i.e.* the last line of (3.18)) which is not modified by gauge fields, and the hyper-multiplet mass has been taken into account. Notice that ϕ and $\bar{\phi}$ do not enter into this action on equal footing. For example only $\bar{\phi}$, and not ϕ, couples to the fermionic colored moduli $\bar{\mu}^A$ and μ^B. This difference has important consequences on the holomorphic structure of the instanton correlators. Actually there are many other non-zero disk diagrams with instanton moduli and gauge fields that survive in the field theory limit. However, as explained in Refs. [8, 18], the corresponding couplings can be easily obtained from those appearing in the action (3.21) by means of supersymmetry Ward identities. In the end, to get the complete expression one has simply to replace all occurrences of the adjoint scalars ϕ and $\bar{\phi}$ with the corresponding $\mathcal{N} = 2$ chiral and anti-chiral superfields, $\widehat{\phi}$ and $\widehat{\bar{\phi}}$. With this understanding, the action (3.21) is then the full result on the Coulomb branch. All other string amplitudes containing more insertions of moduli or gauge field vertices or defined on world-sheets of higher topology, either vanish in the field theory limit or do not contribute in the semi-classical approximation being of higher order in g_a.

The second step is to perform the integration over all moduli of the action (3.21) to obtain the k-instanton induced gauge effective action

$$S_k = \Lambda^{(2N_a - N_F)k} \; e^{\mathcal{A}'_{5a}} \int d^4 x_0 \, d^4\theta \, d\widehat{\mathcal{M}}_k \, e^{-S_{\text{mod}}(\widehat{\phi},\widehat{\bar{\phi}},m;\widehat{\mathcal{M}}_k)} \; . \tag{3.22}$$

A few comments are in order. First of all, even if $S_{\text{mod}}(\widehat{\phi},\widehat{\bar{\phi}},m;\widehat{\mathcal{M}}_k)$ has an explicit dependence on the anti-chiral superfield $\widehat{\bar{\phi}}$, the resulting effective action S_k is a *holomorphic* functional of $\widehat{\phi}$. Indeed, the $\widehat{\bar{\phi}}$ dependence disappears upon integrating over $\widehat{\mathcal{M}}_k$ as a consequence of the cohomology properties of the integration measure on the instanton moduli space [48, 16, 22]. However, to fully specify the holomorphic properties of the instanton induced effective action we have to consider also the contribution of the annulus amplitude that appears in the prefactor of (3.22). In principle this term can introduce a non-holomorphic dependence on the complex and Kähler structure moduli of the compactification space. We will discuss in detail this issue in Sect. 5 after explicitly computing the annulus amplitude for our orbifold compactification in the next section.

It is also worth pointing out that among the centered moduli $\widehat{\mathcal{M}}_k$ there is the singlet part of the anti-chiral fermions $\lambda_{\dot{\alpha}A}$ which is associated to the supersymmetries that are preserved both by the D9 and by the E5 branes. Thus one may naively think that instantonic branes cannot generate an F-term, *i.e.* an integral on half superspace, due to the presence of the anti-chiral $\lambda_{\dot{\alpha}A}$'s among the integration variables. Actually, this is not true since the $\lambda_{\dot{\alpha}A}$'s, including its singlet part, do couple to other instanton moduli (see the last terms in Eq. (3.18)) and their integration can be explicitly performed yielding the fermionic ADHM constraints on the moduli space. Things would be very different instead, if there were no D9$_a$ branes, that is if we were discussing the case of the exotic instantons. In this case, due to the different structure of the charged moduli, the singlet part of the $\lambda_{\dot{\alpha}A}$'s would not couple to anything and, unless it is removed from the spectrum, for example with an orientifold projection [30, 31, 32], an integral like the one in (3.22) would

vanish. In the case of ordinary gauge instantons, instead, we can write

$$S_k = \int d^4x_0 \, d^4\theta \, \mathcal{F}_k(\widehat{\phi}, m) \ , \tag{3.23}$$

where the prepotential

$$\mathcal{F}_k(\widehat{\phi}, m) = \Lambda^{(2N_a - N_F)k} \, e^{\mathcal{A}'_{5a}} \int d\widehat{\mathcal{M}}_k \, e^{-S_{\text{mod}}(\widehat{\phi}, \widehat{\overline{\phi}}, m; \widehat{\mathcal{M}}_k)} \tag{3.24}$$

is the centered instanton partition function in the presence of $\widehat{\phi}$. The integral over $\widehat{\mathcal{M}}_k$ can be performed using localization techniques [57]. Choosing a low-energy profile for the adjoint superfield of the form

$$\widehat{\phi}_{uv} = \widehat{\phi}_u \, \delta_{uv} \tag{3.25}$$

where $u, v = 1, ..., N_a$ and $\sum_u \widehat{\phi}_u = 0$, so that in the effective theory the gauge group $\mathrm{SU}(N_a)$ is generically broken to $\mathrm{U}(1)^{N_a-1}$, the prepotential for $k = 1$ turns out to be

$$\mathcal{F}_1(\widehat{\phi}, m) = \Lambda^{2N_a - N_F} \, e^{\mathcal{A}'_{5a}} \sum_{u=1}^{N_a} \left[\prod_{v \neq u} \frac{1}{(\widehat{\phi}_v - \widehat{\phi}_u)^2} \prod_{f=1}^{N_F} (\widehat{\phi}_u + m_f) \right] . \tag{3.26}$$

Similar closed form expressions can be obtained also for higher values of k (see *e.g.* Ref. [16]). However, for our future considerations the only relevant feature is that the prepotential $\mathcal{F}_k(\widehat{\phi}, m)$ is a homogeneous function of its variables, and specifically

$$\mathcal{F}_k(\xi \, \widehat{\phi}, \xi \, m) = \xi^{2-(2N_a - N_F)k} \, \mathcal{F}_k(\widehat{\phi}, m) \tag{3.27}$$

as one can check from the definition (3.24).

It is also convenient to write the effective action S_k in terms of (abelian) $\mathcal{N} = 1$ superfields, by decomposing the $\mathcal{N} = 2$ superfield $\widehat{\phi}$ into its $\mathcal{N} = 1$ components ϕ and W_α. Then we have

$$S_k = \Lambda^{(2N_a - N_F)k} \, e^{\mathcal{A}'_{5a}} \left\{ \int d^4x_0 \, d^2\theta \left[\frac{1}{2g_a^2} \, \tau_{uv}(\phi, m) \, W_u^\alpha W_{\alpha v} \right] \right.$$
$$\left. + \int d^4x_0 \, d^2\theta \, d^2\bar{\theta} \left[\frac{1}{g_a^2} \, \bar{\phi}_u \Phi_u^{\mathrm{D}}(\phi, m) \right] \right\} , \tag{3.28}$$

where the functions τ and Φ^{D} are defined by

$$g_a^2 \left(\frac{\partial^2 \mathcal{F}_k}{\partial \widehat{\phi}_u \partial \widehat{\phi}_v} \right)_{\widehat{\phi} = \phi} = \Lambda^{(2N_a - N_F)k} \, e^{\mathcal{A}'_{5a}} \, \tau_{uv}(\phi, m) \tag{3.29}$$

and

$$g_a^2 \left(\frac{\partial \mathcal{F}_k}{\partial \widehat{\phi}_u} \right)_{\widehat{\phi} = \phi} = \Lambda^{(2N_a - N_F)k} \, e^{\mathcal{A}'_{5a}} \, \Phi_u^{\mathrm{D}}(\phi, m) . \tag{3.30}$$

All these expressions are written in terms of the canonically normalized fields, but using the homogeneous property (3.27) of the prepotential and the rescalings (2.21) and (2.47), it is straightforward to translate the above result in the supergravity basis, getting

$$S_k = \Lambda^{(2N_a - N_F)k} \, e^{\mathcal{A}'_{5_a}} \quad (g_a \sqrt{K_\Phi})^{(N_F - 2N_a)k} \left\{ \int d^4x_0 \, d^2\theta \left[\frac{1}{2g_a^2} \, \tau_{uv}(\Phi, M) \, W_u^\alpha W_{\alpha v} \right] \right.$$
$$\left. + \int d^4x_0 \, d^2\theta \, d^2\bar\theta \left[K_\Phi \, \bar\Phi_u \Phi_u^{\mathrm{D}}(\Phi, M) \right] \right\} . \tag{3.31}$$

In the following two sections we will carefully analyze the contribution of the annulus amplitude to the prefactor of the effective action S_k and discuss its relevance for the holomorphicity properties of the final expression.

4. The rôle of annulus amplitudes

We now consider in detail the amplitude $\mathcal{A}_{5_a}$ whose "primed" part appears in the prefactor of the non-perturbative effective action. This annulus amplitude represents the 1-loop vacuum energy due to the open strings with at least one end point on the wrapped instantonic branes. Because of supersymmetry, the annulus amplitude associated to the E5$_a$/E5$_a$ strings identically vanishes, so that $\mathcal{A}_{5_a}$ receives contributions only from mixed annuli with one boundary on the E5$_a$'s and the other on the D9 branes. These mixed amplitudes describe the 1-loop contributions of the charged instantonic open strings (*i.e.* the E5$_a$/D9$_a$ and D9$_a$/E5$_a$ strings) and of the flavored instantonic open strings (*i.e.* the E5$_a$/D9$_b$ and D9$_b$/E5$_a$ strings). Their explicit expressions will be determined in Sect. 4.2, but before doing this we present in the next subsection a general argument that explains their meaning and their relation with the running gauge coupling constant.

4.1 The mixed annuli and the running gauge coupling constant

Let us consider the gauge kinetic term at tree level

$$S = \frac{1}{g^2} \int d^4x \, \mathrm{Tr} \left\{ \frac{1}{2} \, F_{\mu\nu}^2 \right\} . \tag{4.1}$$

If we take a constant magnetic field whose only non-zero component is $F_{23} = fT$ where T is a specific generator of the gauge group, then the action (4.1) simply becomes

$$S(f) = \frac{V_4 \, f^2}{2 \, g^2} , \tag{4.2}$$

where V_4 is the (regularized) volume of space-time. On the other hand, if we consider an instanton configuration with charge k, then the classical action (4.1) is

$$S_{\mathrm{inst}} = \frac{8\pi^2 k}{g^2} . \tag{4.3}$$

From these formulas it is immediate to realize that

$$\frac{S_{\mathrm{inst}}}{8\pi^2 k} = \frac{S(f)''}{V_4} , \tag{4.4}$$
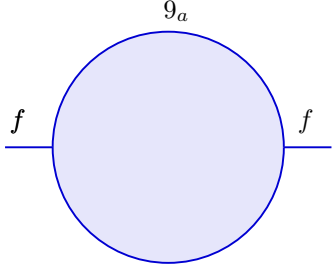
Figure 2: The amplitude $\mathcal{D}_{9_a}(f)$: a disk whose boundary lies on the D9$_a$ branes, with the insertion of the gauge field at the quadratic order.
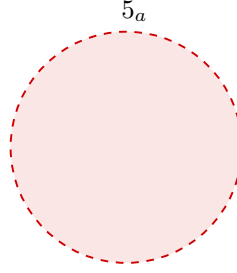
Figure 3: The amplitude $\mathcal{D}_{5_a}$: a disk whose boundary lies on k wrapped E5$_a$ branes.

where $''$ means second derivative with respect to f. Such a relation simply expresses the equality of the gauge coupling constant computed in two different backgrounds.

In the case of supersymmetric theories the same relation (4.4) holds also at the quantum level, after taking into account the 1-loop corrections. In fact, in the constant f background the action (4.2) gets replaced by

$$S(f) + S^{1-\text{loop}}(f) = \frac{V_4 f^2}{2\, g^2(\mu)} \;, \tag{4.5}$$

where $g(\mu)$ is the running coupling constant at scale μ, *i.e.*

$$\frac{1}{g^2(\mu)} = \frac{1}{g^2} + \frac{b_1}{16\pi^2} \log \frac{\mu^2}{\Lambda_{\text{UV}}^2} \tag{4.6}$$

with Λ_{UV} being the ultra-violet cutoff and b_1 the 1-loop coefficient of the β-function. Similarly, if we consider 1-loop fluctuations around the instanton background, the action (4.3) is simply replaced by

$$S_{\text{inst}} + S_{\text{inst}}^{1-\text{loop}} = \frac{8\pi^2 k}{g^2(\mu)} \;. \tag{4.7}$$

Indeed, in a supersymmetric theory the 1-loop determinants of the non-zero-modes fluctuations around the instanton cancel out [59] and the only effect is the renormalization of the gauge coupling constant. Comparing (4.5) and (4.7) we easily see that the same relation (4.4) holds also for the 1-loop corrected actions.

We now show how to rephrase the previous arguments in string theory. As explained in Sect. 2, to obtain $S(f)$ at tree-level we can take a stack of D9$_a$ branes wrapped on a six-torus and compute the DBI action (2.15) in a constant background gauge field, choosing as before $F_{23} = fT$ and then expanding it to quadratic order in f. The result is precisely Eq. (4.2) with the coupling constant g_a given in (2.51). This is equivalent to compute a tree-level amplitude $\mathcal{D}_{9_a}(f)$ described by a disk with two insertions of vertex operators for f along its boundary which lies on the D9$_a$ branes (see Fig. 2). More precisely, in Euclidean signature such amplitude is minus the action $S(f)$, namely

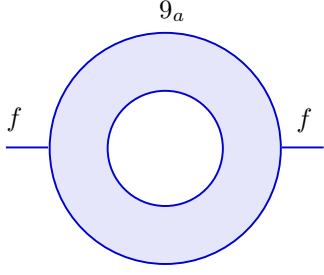$$\mathcal{D}_{9_a}(f) = -\frac{V_4 f^2}{2\, g_a^2} \;. \tag{4.8}$$

Figure 4: The amplitudes $\mathcal{A}_{9_a;9_a}(f)$ or $\mathcal{A}_{9_a;9_b}(f)$ correspond to annuli where one boundary lies on the D9$_a$ branes and carries two insertions of f, while the other boundary lies, respectively, on a D9$_a$ or a D9$_b$ brane.
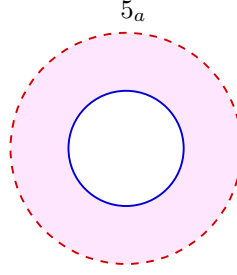
Figure 5: The amplitudes $\mathcal{A}_{5_a;9_a}$ or $\mathcal{A}_{5_a;9_b}$ correspond to annuli where one boundary lies on the E5a branes, while the other boundary lies, respectively, on a D9$_a$ or a D9$_b$ brane.

On the other hand, in our string model the classical instanton action S_{inst} is obtained from the vacuum amplitude $\mathcal{D}_{5_a}$ on a disk whose boundary lies on k wrapped E5$_a$ branes as we already explained in Eq. (3.15), graphically represented in Fig. 3, which we rewrite here for convenience:

$$\mathcal{D}_{5_a} = -\frac{8\pi^2 k}{g_a^2} \ . \tag{4.9}$$

Thus, from (4.8) and (4.9) we straightforwardly obtain a relation between the vacuum disk amplitude with E5$_a$ boundary conditions and the 2-point function on a disk with D9$_a$ boundary conditions, namely

$$\frac{\mathcal{D}_{5_a}}{8\pi^2 k} = \frac{\mathcal{D}_{9_a}(f)''}{V_4} \tag{4.10}$$

in strict analogy with the field theory result (4.4).

The same kind of relation holds also for 1-loop amplitudes. In fact, in the constant gauge field background the 1-loop correction to the classical action is obtained by computing the vacuum amplitude on an annulus[13] with one boundary on the brane with f and the second boundary on the other branes [58, 52], and then expanding the result to second order in f. This is equivalent to compute the 2-point function $\mathcal{A}_{9_a;9_a}(f)$ represented in Fig. 4 where the loop is spanned by the D9$_a$/D9$_a$ strings. If also flavor branes are present, we should consider also the annulus amplitude $\mathcal{A}_{9_a;9_b}(f)$ with D9$_a$/D9$_b$ and D9$_b$/D9$_a$ strings circulating in the loop. These open string amplitudes exhibit both UV and IR divergences. The UV divergences, corresponding to IR divergences in the dual closed string channel, cancel in consistent tadpole-free models; even if in this paper we take only a local point of view, we assume that globally the closed string tadpoles are absent so that we can ignore the UV divergences. On the other hand, introducing a cutoff μ to regulate the IR divergences, the above annulus amplitudes take the following form [49, 52]

$$\mathcal{A}_{9_a}(f) \equiv \mathcal{A}_{9_a;9_a}(f) + \mathcal{A}_{9_a;9_b}(f) = -\frac{V_4 f^2}{2} \left(\frac{b_1}{16\pi^2} \log(\alpha'\mu^2) + \Delta_a \right) \ . \tag{4.11}$$

---

[13]In orientifold models also the Möbius strip has to be considered.

The logarithmic term accounts for the massless open string states circulating in the loop and is thus proportional to the coefficient b_1 of the β-function. The finite term Δ_a originates from the integration over massive states and represents the threshold corrections. In principle, these are due to excited string states and/or to Kaluza-Klein modes arising from the compactification of the six extra dimensions. In $\mathcal{N} = 2$ models, however, only the Kaluza-Klein partners of the massless string states do contribute while the excited string states cancel each other [58, 52]. Notice also that in (4.11) the rôle of the UV cutoff is played naturally by the string length.

Let us now consider the instanton background. As we mentioned above, the 1-loop amplitudes in this case correspond to mixed annulus diagrams with one boundary on the instantonic $E5_a$ branes and the other boundary on the color $D9_a$ branes or on the flavor $D9_b$ branes. These amplitudes, denoted $\mathcal{A}_{5_a;9_a}$ and $\mathcal{A}_{5_a;9_b}$ respectively, are represented in Fig. 5 and will be explicitly computed in the following subsection. However, even before computing them, we can understand their meaning using the relation (4.10) which allows to trade the boundary on the $E5_a$ for a boundary on the $D9_a$'s with a constant field f. Thus, in a supersymmetric model the total annulus amplitude $\mathcal{A}_{5_a}$ must account only for the 1-loop correction to the gauge coupling constant in a k instanton background and, after regulating the IR divergences, we expect to find

$$\mathcal{A}_{5_a} \equiv \mathcal{A}_{5_a;9_a} + \mathcal{A}_{5_a;9_b} = -8\pi^2 k \left(\frac{b_1}{16\pi^2} \log(\alpha'\mu^2) + \Delta_a \right) \ . \tag{4.12}$$

Notice that in this context the β-function coefficient b_1 arises from the counting (with appropriate sign and weight) of the bosonic and fermionic ground states of mixed open strings with one end point on the $E5_a$ branes, *i.e.* from the charged and flavored instanton moduli that we listed in Sect. 3.1. We will elaborate more on this point in the following subsection after the explicit computation of the mixed annulus amplitudes.

From (4.11) and (4.12) it immediately follows that

$$\frac{\mathcal{A}_{5_a}}{8\pi^2 k} = \frac{\mathcal{A}_{9_a}(f)''}{V_4} \ , \tag{4.13}$$

which is the natural generalization of (4.10) at 1-loop. The relation (4.13) between the annulus with a boundary on the instantonic brane and the annulus with a constant gauge field f, which has been noticed in Refs. [26, 27], is the strict analogue of the field theory relation (4.4) and it simply expresses the equality of the (running) gauge coupling constant computed in two different backgrounds. From our arguments it also follows that in supersymmetric models the annulus amplitudes with wrapped Euclidean branes and no moduli insertions, contrarily to some claims in the literature, seem not to be related to the 1-loop determinants of the non-zero mode fluctuations around the instanton background, which in fact are known to exactly cancel out because of supersymmetry [59].

4.2 The explicit form of the annulus amplitude $\mathcal{A}_{5_a}$

The annulus amplitude $\mathcal{A}_{5_a}$ is the 1-loop free energy of the open strings suspended between the $E5_a$ branes and the D9 branes and, as indicated in (4.12), consists of a contribution from

the charged instanton sector, $\mathcal{A}_{5_a;9_a}$, and a contribution from the flavored instanton sector, $\mathcal{A}_{5_a;9_b}$. In turn each of these individual contributions is a sum of two terms corresponding to the two possible orientations of the open strings, $e.g.$

$$\mathcal{A}_{5_a;9_a} \equiv \mathcal{A}(9_a/5_a) + \mathcal{A}(5_a/9_a) , \qquad (4.14)$$

and similarly for the flavored strings. Let us now give some details on these amplitudes, starting from the charged sector.

The charged instanton sector   For a given open string orientation, the annulus amplitude in the charged sector has the following schematic form

$$\mathcal{A}(9_a/5_a) = \int_0^\infty \frac{d\tau}{2\tau}\Big[\mathrm{Tr}_{\,\mathrm{NS}} \left(P_{\mathrm{GSO}}^{(9_a/5_a)} \, P_{\mathrm{orb}} \, q^{L_0} \right) - \mathrm{Tr}_{\,\mathrm{R}} \left(P_{\mathrm{GSO}}^{(9_a/5_a)} \, P_{\mathrm{orb}} \, q^{L_0} \right)\Big] , \qquad (4.15)$$

where

$$P_{\mathrm{GSO}}^{(9_a/5_a)} = \frac{1 + (-1)^F}{2} \qquad (4.16)$$

is the GSO projector, P_{orb} is the orbifold projector (see Eq. (2.11)) and $q = \exp(-2\pi\tau)$. The amplitude $\mathcal{A}(5_a/9_a)$ corresponding to open strings with opposite orientation is analogous to (4.15), but we must consider the possibility that the GSO projection $P_{\mathrm{GSO}}^{(5_a/9_a)}$ to be employed in this case may be different from the one in (4.16). We will argue that this is indeed the case in the R sector.

The traces in (4.15) are taken over the states in the CFT of the open strings with D9$_a$/E5$_a$ boundary conditions and over the Chan-Paton indices as well. The CFT contains various components: the string fields X^μ and ψ^μ in the space-time directions, those along the orbifold $(\mathcal{T}_2^{(1)} \times \mathcal{T}_2^{(2)})/\mathbb{Z}_2$, those along $\mathcal{T}_2^{(3)}$ and the ghost/superghost system. Let us now discuss briefly the contributions of these various components.

The fields X^μ and ψ^μ in the space-time directions have Neumann-Dirichlet conditions, as discussed in Sect. 3, and hence are twisted by $1/2$; in particular, their contribution to the trace in the NS$(-1)^F$ structure vanishes because of the fermionic zero-modes.

Moving to the internal directions, all fields Z^i and Ψ^i have Neumann-Neumann boundary conditions and thus are untwisted, but for $i = 1,2$ they are reflected by the $\mathbb{Z}_2$ action so that they yield different non-zero mode contributions depending on whether the orbifold generator h is inserted or not in the trace. On the other hand, the fields Z^3 and Ψ^3 are not acted upon by the orbifold and their non-zero mode contributions cancel exactly against those from the ghost/superghost system. Concerning the zero-modes, the trace over the discretized momenta of the bosonic fields Z^i gives a contribution of the form $\mathcal{Y}^{(1)}\mathcal{Y}^{(2)}\mathcal{Y}^{(3)}$ where

$$\mathcal{Y}^{(i)} \equiv \sum_{(r_1,r_2)\in\mathbb{Z}^2} q^{\frac{r_p}{n_a^{(i)}} \mathcal{G}_{(i)}^{pq} \frac{r_q}{n_a^{(i)}}} = \sum_{(r_1,r_2)\in\mathbb{Z}^2} q^{\frac{T_2^{(i)}}{U_2^{(i)}} \frac{|r_1 U^{(i)} - r_2|^2}{|n_a^{(i)} T^{(i)} - m_a^{(i)}|^2}} = \sum_{(r_1,r_2)\in\mathbb{Z}^2} q^{\frac{|r_1 U^{(i)} - r_2|^2}{U_2^{(i)} T_2^{(i)} |\ell_a^{(i)}|^2}} . \quad (4.17)$$

In this expression $\mathcal{G}_{(i)}^{pq}$ is the inverse open string metric on the i-th torus[14] and in the last step we have used the definition (2.52). Notice however that when h is inserted in the

---

[14]The open string is defined as $\mathcal{G}_{(i)} = \left(G_{(i)} + B_{(i)} - 2\pi\alpha' F_{(i)}\right) G_{(i)}^{-1} \left(G_{(i)} - B_{(i)} + 2\pi\alpha' F_{(i)}\right).$

trace, only the zero-momentum states along the first two tori survive and thus in this case the bosonic zero-mode contribution reduces to $\mathcal{Y}^{(3)}$.

Finally, let us consider the fermionic zero modes of the Ψ^i fields in the R sector. There are eight zero modes corresponding to the following states:

$$
\begin{aligned}
|\Delta S_A\rangle &\equiv \left\{ |\Delta S_{-++}\rangle, |\Delta S_{+-+}\rangle, |\Delta S_{++-}\rangle, |\Delta S_{---}\rangle \right\}, \\
|\Delta S^A\rangle &\equiv \left\{ |\Delta S^{+--}\rangle, |\Delta S^{-+-}\rangle, |\Delta S^{--+}\rangle, |\Delta S^{+++}\rangle \right\},
\end{aligned}
\tag{4.18}
$$

where S^A and S_A are the spin fields in the six-dimensional internal space. The action of $(-1)^F$ on these states is defined to be

$$
(-1)^F |\Delta S_A\rangle = +|\Delta S_A\rangle \quad , \quad (-1)^F |\Delta S^A\rangle = -|\Delta S^A\rangle,
\tag{4.19}
$$

while the orbifold action is given in (2.10). Thus the GSO projection (4.16) selects the states $|\Delta S_A\rangle$ that are associated to four charged fermionic moduli μ^A, of which only two are h-invariant and appear in the physical spectrum of the $D9_a/E5_a$ strings, as described in Sec. 3.1. With this information it is possible to evaluate in a straightforward manner the contribution of these fermionic zero-modes to the trace in the 1-loop amplitude. In the odd spin structure, because of the insertion of $(-1)^F$, this trace vanishes but simultaneously the superghost zero-modes give a divergent contribution, which makes the entire expression ill-defined. However, as discussed in Ref. [60], there exists a suitable regularization procedure for both contributions which makes their product well-defined and actually finite. In particular, it turns out (see for example the discussion after Eq. (B.11) of Ref. [61]) that the trace over the fermionic zero-modes vanishes when we insert $(-1)^F$ or h, while it equals $8/2 = 4$ when there is no insertion or the insertion of $(-1)^F h$.

Altogether, collecting all the previous information, we can obtain the explicit expression for the amplitude $\mathcal{A}(9_a/5_a)$. In the NS spin structure we find

$$
\begin{aligned}
\mathcal{A}(9_a/5_a)_{\text{NS}} &\equiv \frac{1}{2} \int_0^\infty \frac{d\tau}{2\tau} \, \text{Tr}_{\text{NS}} \left(P_{\text{orb}} \, q^{L_0} \right) \\
&= \frac{N_a k}{2} \int_0^\infty \frac{d\tau}{2\tau} \left[\frac{1}{2} \left(\frac{\theta_2(0)^2 \, \theta_3(0)^2}{\theta_4(0)^2 \, \theta_1'(0)^2} \, \mathcal{Y}^{(1)} \mathcal{Y}^{(2)} \mathcal{Y}^{(3)} + 4 \, \mathcal{Y}^{(3)} \right) \right],
\end{aligned}
\tag{4.20}
$$

where the θ_a's are the Jacobi θ-functions (we follow the conventions of Appendix A of Ref. [52]). The second term in (4.20) contains the insertion of h, upon which the non-zero modes contributions along the orbifold and the space-time directions cancel each other. The factor of $1/2$ inside the square bracket comes from the orbifold projector. As argued above, the NS$(-1)^F$ structure vanishes identically. In the R sector, taking into account the minus sign due to spin-statistics, we get

$$
\begin{aligned}
\mathcal{A}(9_a/5_a)_{\text{R}} &\equiv -\frac{1}{2} \int_0^\infty \frac{d\tau}{2\tau} \, \text{Tr}_{\text{R}} \left(P_{\text{orb}} \, q^{L_0} \right) \\
&= -\frac{N_a k}{2} \int_0^\infty \frac{d\tau}{2\tau} \left[\frac{1}{2} \left(\frac{\theta_3(0)^2 \, \theta_2(0)^2}{\theta_4(0)^2 \, \theta_1'(0)^2} \, \mathcal{Y}^{(1)} \mathcal{Y}^{(2)} \mathcal{Y}^{(3)} \right) \right],
\end{aligned}
\tag{4.21}
$$

which comes entirely from the term with no insertion of h. Finally, the odd spin structure $\mathrm{R}(-1)^F$ receives a contribution only when h is inserted, and reads

$$\begin{aligned}
\mathcal{A}(9_a/5_a)_{\mathrm{R}(-1)^F} &\equiv -\frac{1}{2}\int_0^\infty \frac{d\tau}{2\tau}\,\mathrm{Tr}_{\,\mathrm{R}}\left((-1)^F\,P_{\mathrm{orb}}\,q^{L_0}\right)\\
&= -\frac{N_a k}{2}\int_0^\infty \frac{d\tau}{2\tau}\Big[\frac{1}{2}\left(4\,\mathcal{Y}^{(3)}\right)\Big]\ .
\end{aligned}\tag{4.22}$$

Then the full GSO-projected amplitude for the D9$_a$/E5$_a$ strings is

$$\mathcal{A}(9_a/5_a) = \mathcal{A}(9_a/5_a)_{\mathrm{NS}} + \mathcal{A}(9_a/5_a)_{\mathrm{R}} + \mathcal{A}(9_a/5_a)_{\mathrm{R}(-1)^F} = 0\ ,\tag{4.23}$$

where we have inserted the results (4.20), (4.21) and (4.22).

However, we have to consider also the amplitude $\mathcal{A}(5_a/9_a)$ which is the 1-loop vacuum energy of open strings with the opposite orientation. The only subtlety occurs in the R sector. In this case we have again eight fermionic ground states, namely $|\bar{\Delta}S_A\rangle$ and $|\bar{\Delta}S^A\rangle$ which differ from the states (4.18) only because they contain the anti-twist $\bar{\Delta}$ in place of Δ. The $(-1)^F$ parity on these states must be defined consistently with the previous definition (4.19). To do so, let us observe that

$$\langle\,\bar{\Delta}S^A\,|\,\Delta S_B\,\rangle = \delta^A_B\ .\tag{4.24}$$

This pairing, together with (4.19), implies the following parity assignments

$$(-1)^F\,|\bar{\Delta}S_A\rangle = -\,|\bar{\Delta}S_A\rangle\ ,\quad (-1)^F\,|\bar{\Delta}S^A\rangle = +\,|\bar{\Delta}S^A\rangle\ .\tag{4.25}$$

As discussed after Eq. (3.7), the physical spectrum of the $5_a/9_a$ strings contains the moduli $\bar{\mu}^A$ with the *same* chirality as the μ^A. Thus, the GSO projection must select the corresponding states, namely $|\bar{\Delta}S_A\rangle$ which are odd under $(-1)^F$. Therefore, in the R sector of the $5_a/9_a$ strings we must take

$$P_{\mathrm{GSO}}^{(5_a/9_a)} = \frac{1-(-1)^F}{2}\ .\tag{4.26}$$

as opposed to (4.16). The full GSO-projected amplitude is then

$$\mathcal{A}(5_a/9_a) = \mathcal{A}(5_a/9_a)_{\mathrm{NS}} + \mathcal{A}(5_a/9_a)_{\mathrm{R}} - \mathcal{A}(5_a/9_a)_{\mathrm{R}(-1)^F}\tag{4.27}$$

with a crucial minus sign in the odd spin structure as compared to (4.23). The individual terms in this expression can be computed as explained above and turn out to be equal to the corresponding ones for the other orientation, given in Eqs. (4.20), (4.21) and (4.22) respectively. Now, however, due to the different sign in the $\mathrm{R}(-1)^F$ sector, the amplitude $\mathcal{A}(5_a/9_a)$ is not zero. The fact that the annulus amplitude is different for the two open string orientations when the odd spin structure is non zero should not come as a surprise; in fact the same thing has been noticed in other systems with similar features, most notably in the D0/D8 brane systems or their T-duals [60].

From the above analysis we conclude that the total amplitude (4.14) is

$$\mathcal{A}_{5_a;9_a} = \mathcal{A}(5_a/9_a) = 2N_a k\int_0^\infty \frac{d\tau}{2\tau}\,\mathcal{Y}^{(3)}\ .\tag{4.28}$$

In the end only the zero-modes contribute to this annulus amplitude: they correspond to the charged instanton moduli listed in Sect. 3.1, and their Kaluza-Klein partners on the torus $\mathcal{T}_2^{(3)}$ that together reconstruct the sum in $\mathcal{Y}^{(3)}$.

The flavored instanton sector   Let us now consider the annulus amplitude produced by the instantonic strings stretching between the flavor D9$_b$ branes and the E5$_a$'s, namely

$$\mathcal{A}_{5_a;9_b} \equiv \mathcal{A}(9_b/5_a) + \mathcal{A}(5_a/9_b) \ . \tag{4.29}$$

The difference with the charged case considered above resides entirely in the CFT of the string fields Z^i and Ψ^i, with $i = 1, 2$, along the orbifold. These fields are all twisted by the same angle $\nu_{ba}^{(1)} = \nu_{ba}^{(2)}$, which in the following will be simply denoted by ν, and none of them has zero-modes. We have however to include the factor I_{ba}, defined in Eq. (2.60), related to the number of Landau levels for these magnetized directions. Another difference is that the fractional branes of type a and b belong to different irreducible representations of the $\mathbb{Z}_2$ orbifold group, so that h acts on the Chan-Paton factors of the open strings as a minus sign, and the twisted NS ground state is h-even while the twisted R ground state is h-odd as explained in Sect. 2.2.

Taking all these facts into account, we can write the various contributions to the annulus amplitude. Let us start with the D9$_b$/E5$_a$ orientation. In the NS spin structure we have

$$
\begin{aligned}
\mathcal{A}(9_b/5_a)_{\text{NS}} &\equiv \frac{1}{2} \int_0^\infty \frac{d\tau}{2\tau} \, \text{Tr}_{\text{NS}} \left(P_{\text{orb}} \, q^{L_0} \right) \\
&= -\frac{N_b I_{ba} k}{2} \int_0^\infty \frac{d\tau}{2\tau} \left[\frac{1}{2} \left(\frac{\theta_2(0)^2 \, \theta_3(i\nu\tau)^2}{\theta_4(0)^2 \, \theta_1(i\nu\tau)^2} + \frac{\theta_2(0)^2 \, \theta_4(i\nu\tau)^2}{\theta_4(0)^2 \, \theta_2(i\nu\tau)^2} \right) \mathcal{Y}^{(3)} \right] \ .
\end{aligned}
\tag{4.30}
$$

The NS$(-1)^F$ amplitude vanishes because of the space-time fermion zero modes, as before. In the R sector we find instead

$$
\begin{aligned}
\mathcal{A}(9_b/5_a)_{\text{R}} &\equiv -\frac{1}{2} \int_0^\infty \frac{d\tau}{2\tau} \, \text{Tr}_{\text{R}} \left(P_{\text{orb}} \, q^{L_0} \right) \\
&= \frac{N_b I_{ba} k}{2} \int_0^\infty \frac{d\tau}{2\tau} \left[\frac{1}{2} \left(\frac{\theta_3(0)^2 \, \theta_2(i\nu\tau)^2}{\theta_4(0)^2 \, \theta_1(i\nu\tau)^2} + \frac{\theta_3(0)^2 \, \theta_1(i\nu\tau)^2}{\theta_4(0)^2 \, \theta_2(i\nu\tau)^2} \right) \mathcal{Y}^{(3)} \right] \ .
\end{aligned}
\tag{4.31}
$$

Finally, the R$(-1)^F$ amplitude, to which both the term without h and the one with h contribute, is

$$\mathcal{A}(9_b/5_a)_{\text{R}(-1)^F} \equiv -\frac{1}{2} \int_0^\infty \frac{d\tau}{2\tau} \, \text{Tr}_{\text{R}} \left((-1)^F \, P_{\text{orb}} \, q^{L_0} \right) = \frac{N_b I_{ba} k}{2} \int_0^\infty \frac{d\tau}{2\tau} \, \mathcal{Y}^{(3)} \ . \tag{4.32}$$

Using the Riemann identities

$$
\begin{aligned}
\theta_2(0)^2 \, \theta_3(i\nu\tau)^2 - \theta_3(0)^2 \, \theta_2(i\nu\tau)^2 &= \theta_4(0)^2 \, \theta_1(i\nu\tau)^2 \ , \\
\theta_2(0)^2 \, \theta_4(i\nu\tau)^2 - \theta_3(0)^2 \, \theta_1(i\nu\tau)^2 &= \theta_4(0)^2 \, \theta_2(i\nu\tau)^2 \ ,
\end{aligned}
\tag{4.33}
$$

one can easily see that the GSO projected amplitude vanishes:

$$\mathcal{A}(9_b/5_a) = \mathcal{A}(9_b/5_a)_{\text{NS}} + \mathcal{A}(9_b/5_a)_{\text{R}} + \mathcal{A}(9_b/5_a)_{\text{R}(-1)^F} = 0 \ . \tag{4.34}$$

Let us now consider the amplitude $\mathcal{A}(5_a/9_b)$ corresponding to the other orientation. Just as in the charged case previously discussed, we must be careful with the GSO projection in the R sector. The same argument presented above implies that $P_{\mathrm{GSO}}^{(5a/9b)}$ and $P_{\mathrm{GSO}}^{(9b/5a)}$ must be defined in a different way. Indeed, the physical states of the two types of strings are described by the vertex operators (3.8) and (3.9) which both contain the same spin field S_- in the last complex direction. Thus, if the μ''s are even under $(-1)^F$, the $\bar{\mu}'$'s, which do not contain the conjugate spin field, must be odd under $(-1)^F$. Then, the complete GSO projected amplitude for the $\mathrm{E}5_a/\mathrm{D}9_b$ strings reads

$$\mathcal{A}(5_a/9_b) = \mathcal{A}(5_a/9_b)_{\mathrm{NS}} + \mathcal{A}(5_a/9_b)_{\mathrm{R}} - \mathcal{A}(5_a/9_b)_{\mathrm{R}(-1)^F} . \tag{4.35}$$

The individual contributions can be computed explicitly as before; one simply has to replace $\nu \to (1 - \nu)$, which however has no consequences because of the properties of the θ-functions, and also change the prefactor to $N_b I_{ab}$, which is also harmless since $I_{ab} = I_{ba}$, see Eq. (2.60). Thus, the various terms in (4.35) are equal to the corresponding ones for the other orientation, given respectively in Eqs. (4.30), (4.31) and (4.32). Now, however, due to the minus sign in the odd spin structure, the amplitude $\mathcal{A}(5_a/9_b)$ is not vanishing.

We thus conclude that the total instantonic amplitude in the flavored sector is

$$\mathcal{A}_{5_a;9_b} = \mathcal{A}(5_a/9_b) = -N_F\, k \int_0^\infty \frac{d\tau}{2\tau}\, \mathcal{Y}^{(3)} , \tag{4.36}$$

where N_F is the number of flavors defined in (2.59).

The total amplitude Summing the contributions (4.28) and (4.36) of the charged and flavor sectors, we finally have

$$\mathcal{A}_{5_a} = (2N_a - N_F)\, k_a \int_0^\infty \frac{d\tau}{2\tau}\, \mathcal{Y}^{(3)} . \tag{4.37}$$

This amplitude is proportional to the 1-loop coefficient b_1 of the β-function of our $\mathcal{N} = 2$ theory, i.e. $b_1 = 2N_a - N_F$. It is interesting to notice that in this context this coefficient arises from the counting of the charged and flavored zero-modes of the instantonic strings. Let us consider in more detail this contribution, tracing back the NS and R terms and keeping them distinct. We have

$$\int_0^\infty \frac{d\tau}{2\tau}\Big[(4N_a - 2N_F)k - (2N_a - N_F)k\Big] = \Big(n_{\mathrm{bos}} - \frac{1}{2}\, n_{\mathrm{ferm}}\Big) \int_0^\infty \frac{d\tau}{2\tau} , \tag{4.38}$$

where

$$n_{\mathrm{bos}} = n_{\mathrm{ferm}} = (4N_a - 2N_F)k \tag{4.39}$$

is the number of bosonic and fermionic moduli in the charged and flavored sectors, i.e. the number of w's and $\bar{w}$'s and the number of μ's, $\bar{\mu}$'s, μ''s and $\bar{\mu}'$'s. Notice also that the stringy origin of the factor of $1/2$ in (4.38) is in the (regularized) trace over the superghost zero-modes of the R sector [60].

To obtain the explicit expression of the annulus amplitude $\mathcal{A}_{5_a}$ we have to compute the integral

$$I \equiv \int_0^\infty \frac{d\tau}{\tau} \mathcal{Y}^{(3)} = \int_0^\infty \frac{d\tau}{\tau} \sum_{(r_1,r_2)\in\mathbb{Z}^2} \mathrm{e}^{-2\pi\tau \frac{|r_1 U^{(3)} - r_2|^2}{U_2^{(3)} T_2^{(3)} |\ell_a^{(3)}|^2}} \ . \tag{4.40}$$

The detailed calculation is performed in Appendix A; here we simply recall that this integral is divergent both in the UV limit $\tau \to 0$, and in the IR limit $\tau \to \infty$. The UV divergence can be reinterpreted as an IR divergence in the dual closed string channel after Poisson resummation. We assume that such a divergence cancels in fully consistent models which satisfy the tadpole cancellation condition [49]. Subtracting this divergence, the integral I can be evaluated by introducing a mass parameter m which regularizes the IR singularity in the open string channel, and the final result is (see Eq. (A.10))

$$I = -\log\left(\alpha' m^2\right) - \log|\eta(U^{(3)})|^4 - \log\left(U_2^{(3)} T_2^{(3)} |\ell_a^{(3)}|^2\right) \ , \tag{4.41}$$

where η is the Dedekind function. Since only one of the two orientations contributes to $\mathcal{A}_{5_a}$, it is possible, following Refs. [58, 52], to take a complex IR cutoff[15]

$$m = \mu\, \mathrm{e}^{\mathrm{i}\varphi} \ , \tag{4.42}$$

so that the instantonic annulus amplitude becomes

$$\mathcal{A}_{5_a} = -b_1\, k\left(\frac{1}{2}\log(\alpha'\mu^2) + \mathrm{i}\,\varphi + \log|\eta(U^{(3)})|^2 + \frac{1}{2}\log\left(U_2^{(3)} T_2^{(3)} |\ell_a^{(3)}|^2\right)\right) \tag{4.43}$$

and has the expected form (4.12).

5. The holomorphic life of the D-brane instantons

In this section we combine the result we have just obtained for the annulus amplitude with what we have discussed in Sect. 3.3 in order to get the instanton induced corrections to the low-energy effective action of our $\mathcal{N} = 2$ theory.

To this aim, let us first recall that what enters in the instanton calculus is *not* the complete annulus amplitude $\mathcal{A}_{5_a}$, but rather its "primed" part $\mathcal{A}'_{5_a}$. This is obtained from $\mathcal{A}_{5_a}$ by subtracting the logarithmically divergent contribution of the zero-modes to avoid double counting since the integral over them is separately performed in an explicit way [24, 27]. However, as remarked already in Refs. [39, 40], the UV cutoff that one uses in the field theory analysis of a string model is the four-dimensional Planck mass M_P, which is related to α' in the following way:

$$M_P^2 = \frac{1}{\alpha'}\, \mathrm{e}^{-\phi_{10}}\, s_2 \ , \tag{5.1}$$

---

[15]Notice that in general one should regulate, for consistency, the contributions of the two orientations with complex conjugate cutoffs [52].

where ϕ_{10} is the ten-dimensional dilaton. This means that what we have to subtract from $\mathcal{A}_{5_a}$ in order to remove the field theory zero modes contribution is not exactly the $\log(\alpha'\mu^2)$ term. Rather, we have to write

$$\mathcal{A}_{5_a} = -8\pi^2 k \left(\frac{b_1}{16\pi^2} \log \frac{\mu^2}{M_P^2} + \widetilde{\Delta}_a \right) \tag{5.2}$$

with

$$\widetilde{\Delta}_a = \frac{b_1}{8\pi^2} \left(\mathrm{i}\,\varphi + \log |\eta(U^{(3)})|^2 + \frac{1}{2} \log(\mathrm{e}^{-\phi_{10}}\,s_2) + \frac{1}{2} \log(U_2^{(3)} T_2^{(3)}\,|\ell_a^{(3)}|^2) \right) . \tag{5.3}$$

Now the logarithmic term in (5.2) correctly accounts for field theory zero-mode contribution and the remaining finite term is the "primed" part of the annulus contribution that appears in the instantonic amplitudes, namely

$$\mathcal{A}'_{5_a} = -8\pi^2 k\, \widetilde{\Delta}_a . \tag{5.4}$$

To discuss the holomorphic properties of the k-instanton induced effective action we have to first rewrite the above expression in terms of the supergravity variables (2.5), getting

$$\begin{aligned} \mathcal{A}'_{5_a} &= -b_1\,k \left(\mathrm{i}\,\varphi + \log |\eta(u^{(3)})|^2 + \frac{1}{2} \log(s_2) + \log(u_2^{(3)} t_2^{(3)}\,|\ell_a^{(3)}|^2) \right) \\ &= -(2N_a - N_F)\,k \left(\mathrm{i}\,\varphi + \log |\eta(u^{(3)})|^2 - \frac{1}{2} \log(g_a^2) - \frac{1}{2} \log K_\Phi \right) , \end{aligned} \tag{5.5}$$

where in the second line we have made use of Eqs. (2.51) and (2.20). Thus, the part of the prefactor in the instanton amplitudes that comes from the annulus diagrams is

$$\mathrm{e}^{\mathcal{A}'_{5_a}} = \left(|\eta(u^{(3)})|^2\,\mathrm{e}^{\mathrm{i}\,\varphi} \right)^{-(2N_a - N_F)k} (g_a\,\sqrt{K_\Phi})^{(2N_a - N_F)k} . \tag{5.6}$$

This is one of the main results in this paper. It shows that the non holomorphic terms produced by the instanton annulus amplitudes nicely combine in the Kähler metric of the adjoint fields (see also Ref. [33]) and precisely cancel the prefactor $(g_a\,\sqrt{K_\Phi})^{(N_F - 2N_a)k}$ in the non-perturbative effective action (3.31) which is produced by the rescaling from the string basis to the supergravity basis.

Furthermore, by tuning the (arbitrary) phase φ of the IR cutoff to be $\arg\left(\eta(u^{(3)})^2\right)$, we can promote the harmonic term $|\eta(u^{(3)})|^2$ to a purely holomorphic one $\eta(u^{(3)})^2$. Thus, the k-instanton induced effective action (3.31) acquires its final form

$$\begin{aligned} S_k = \Lambda'^{(2N_a - N_F)k} \Bigg\{ &\int d^4 x_0\, d^2\theta\, \left[\frac{1}{2g_a^2}\, \tau_{uv}(\Phi, M)\, W_u^\alpha W_{\alpha v} \right] \\ &+ \int d^4 x_0\, d^2\theta\, d^2\bar{\theta} \left[K_\Phi\, \bar{\Phi}_u \Phi_u^D(\Phi, M) \right] \Bigg\} , \end{aligned} \tag{5.7}$$

where we have performed the rescaling

$$\Lambda' = \Lambda\,\eta(u^{(3)})^{-2} \tag{5.8}$$

which is equivalent to the following holomorphic redefinition of the Wilsonian Yang-Mills coupling constant:

$$\tau_{\mathrm{YM}} \equiv \left(\frac{\theta_{\mathrm{YM}}}{2\pi} + \mathrm{i}\,\frac{4\pi^2}{g_a^2} \right) \;\to\; \tau_{\mathrm{YM}} + \mathrm{i}\,\frac{(2N_a - N_F)}{2\pi}\,\log(\eta(u^{(3)}))^2 \;. \tag{5.9}$$

The effective action (5.7) has the holomorphic structure required by supersymmetry in Wilsonian actions [39, 40]. This result is also a confirmation of the Kähler metrics (2.55) and (2.56) for the adjoint and flavored fields.

6. Conclusions

The detailed analysis of the previous sections shows that the instantonic annulus amplitudes have the right structure to reproduce the appropriate Kähler metric dependence in such a way that the instanton induced effective action becomes purely holomorphic in the variables of the supergravity basis. To further elaborate on this point, it is instructive to consider separately the charged and flavored 1-loop amplitudes $\mathcal{A}_{5_a;9_a}$ and $\mathcal{A}_{5_a;9_b}$, given respectively in Eqs. (4.28) and (4.36), and rewrite them in terms of the Kähler metrics K_Φ and K_Q of the adjoint and fundamental chiral multiplets. Using (2.55) and (2.56), as well as the coupling constant (2.51) and the bulk Kähler potential (2.6), we easily find

$$\mathcal{A}_{5_a;9_a} = -N_a\,k\left(\log\frac{\mu^2}{M_P^2} + \log\left(\eta(u^{(3)})\right)^4 - \log(g_a^2) - \log K_\Phi \right) \;, \tag{6.1}$$

$$\mathcal{A}_{5_a;9_b} = \frac{N_F\,k}{2}\left(\log\frac{\mu^2}{M_P^2} + \log\left(\eta(u^{(3)})\right)^4 - K + 2\log K_Q \right) \;, \tag{6.2}$$

where the phase of the complex IR cutoff has been chosen as discussed in the previous section. These two formulas are particular cases of the expression of the one-loop running coupling constant $g^2(\mu)$ given in [39, 40, 41]. This expression can be written in terms of the corresponding one-loop amplitude $\mathcal{A}$, according to the discussion in section 4.1[16], as follows

$$\mathcal{A} = k\left[-\frac{b}{2}\,\log\frac{\mu^2}{M_P^2} + f + \frac{c}{2}\,K - T(G)\,\log\left(\frac{1}{g^2}\right) + \sum_r n_r\,T(r)\,\log K_r \right] \;, \tag{6.3}$$

where f is a holomorphic quantity, K is the bulk Kähler potential and

$$T(r)\,\delta_{AB} = \mathrm{Tr}_r\big(T_A T_B\big) \;, \quad T(G) = T(\mathrm{adj}) \;,$$
$$b = 3\,T(G) - \sum_r n_r\,T(r) \;, \quad c = T(G) - \sum_r n_r\,T(r) \;, \tag{6.4}$$

with T^A being the generators of the gauge group G and n_r the number of $\mathcal{N} = 1$ chiral multiplets in representation r, having Kähler metric K_r. In fact, the charged annulus amplitude (6.1) corresponds to the case of the adjoint matter ($b = 2N_a$, $c = 0$) while the

---

[16]One has to use the fact that that $1/g^2 = -\mathrm{Re}(\mathcal{A})/8\pi^2 k$.

flavored amplitude (6.2) corresponds to $2N_F$ chiral multiplets in the fundamental representation ($b = c = -N_F$). In both cases, f is proportional to $\log(\eta(u^{(3)})^2$ and represents a finite holomorphic renormalization of the Wilsonian Yang-Mills coupling.

Even if we have considered models with $\mathcal{N} = 2$ supersymmetry, throughout this paper we have mostly used a $\mathcal{N} = 1$ notation, and also Eqs. (6.1) - (6.3) have been written in this language. However, it is not difficult to convert them to a full-fledged $\mathcal{N} = 2$ notation. To this aim, let us observe that in (6.2) the dependence on $t_2^{(1)}$, $t_2^{(2)}$, $u_2^{(1)}$ and $u_2^{(2)}$ actually drops out, so that we can express the result in terms of the $\mathcal{N} = 2$ bulk Kähler potential [55]

$$\widetilde{K} = K - 2\log K_Q = -\log(s_2) - \log(t_2^{(3)}) - \log(u_2^{(3)}) \tag{6.5}$$

without introducing a Kähler metric for the hyper-multiplets. In this way we see that both Eqs. (6.1) and (6.2) are two particular cases of the formula [39, 40, 41, 47]

$$\mathcal{A} = k \left[-\frac{b}{2} \log \frac{\mu^2}{M_P^2} + f - T(G) \log\left(\frac{1}{g^2}\right) + T(G) \log(K_\Phi) - \sum_r N_r T(r) \widetilde{K} \right] , \tag{6.6}$$

where b is again the coefficient of the β-function and N_r is the number of $\mathcal{N} = 2$ hyper-multiplets in the representation r. Notice also that in terms of the Kähler potential (6.5), Eq. (2.44) can be written as

$$\mathrm{e}^{\widetilde{K}/2} = g_a \sqrt{K_\Phi} , \tag{6.7}$$

while Eq. (6.6) becomes

$$\mathcal{A} = k \left[f + \frac{b}{2}\left(\log \frac{M_P^2}{\mu^2} + \widetilde{K} \right) \right] , \tag{6.8}$$

which are in the appropriate form required by $\mathcal{N} = 2$ supergravity [47].

We conclude by stressing that the general formula (6.3) allows to obtain the explicit expression of the Kähler metrics K_r starting from an instantonic annulus amplitude $\mathcal{A}$ in a gauge theory with a specified matter content. This can be particularly useful in the case of $\mathcal{N} = 1$ models in which the Kähler metric of flavored chiral multiplets is not known *a priori* since they correspond to string excitations of twisted sectors. Applying the formula (6.3) to $\mathcal{N} = 1$ theories and using it to check the holomorphicity of the non-perturbative superpotential terms induced by instantons will therefore provide a way to deduce the Kähler metric for the twisted matter in $\mathcal{N} = 1$ theories. This will be the subject of a separate publication [62].

Acknowledgments

A. Calculation of the integral I

In this appendix we give some details on the explicit calculation of the integral

$$I \equiv \int_0^\infty \frac{d\tau}{\tau} \sum_{(r_1, r_2) \in \mathbb{Z}^2} e^{-2\pi\tau \frac{|r_1 U^{(3)} - r_2|^2}{U_2^{(3)} T_2^{(3)} |\ell_a^{(3)}|^2}} . \tag{A.1}$$

Regularization with a cut-off In the IR region ($\tau \to \infty$) the integral (A.1) has a logarithmic divergence due to the massless states and a regularization procedure is necessary to cure the IR problem. Here we use the regularization procedure introduced in Ref. [63] and insert in the integrand the regulator

$$R(\tau) = 1 - e^{-\frac{\pi}{\alpha' m^2 \tau}} , \tag{A.2}$$

where m is a (complex) IR cut-off. In the following we will briefly discuss another regularization scheme with Wilson lines.

Eq. (A.1) is divergent also in the UV-region $\tau \to 0$. This divergence was not present in Ref. [63] and in order to cure it we use the Poisson resummation formula to rewrite Eq. (A.1) in the form:

$$I \equiv \frac{|\ell_a^{(3)}|^2 T_2^{(3)}}{2} \int_0^\infty \frac{d\tau}{\tau^2} \sum_{(s_1, s_2) \in \mathbb{Z}^2 - \{(0,0)\}} e^{-\frac{\pi}{2\tau} \frac{|\ell_a^{(3)}|^2 T^{(3)}}{U_2^{(3)}} |U^{(3)} s_1 + s_2|^2} \left(1 - e^{-\frac{\pi}{\alpha' m^2 \tau}}\right) , \tag{A.3}$$

where we have neglected the divergent contribution due to the term $s_1 = s_2 = 0$, because it is absent in a consistent model free of tadpoles [49].

We can now perform the integral getting:

$$I = \frac{U_2^{(3)}}{\pi} \sum_{(s_1, s_2) \in \mathbb{Z}^2 - \{(0,0)\}} \left[\frac{1}{|U^{(3)} s_1 + s_2|^2} - \frac{1}{|U^{(3)} s_1 + s_2|^2 + U_2^{(3)} N} \right] \tag{A.4}$$

with $N = 2/(\alpha' m^2 T_2^{(3)} |\ell_a^{(3)}|^2)$. By using the identity:

$$\sum_{s_2 \in \mathbb{Z}} \frac{1}{(s_2 + A)^2 + B^2} = \frac{i\pi}{2B} [\cot \pi(A + iB) - \cot \pi(A - iB)]$$

$$= -\frac{\pi}{B} \left[\frac{e^{2\pi i u}}{e^{2\pi i u} - 1} + \frac{e^{-2\pi i \bar{u}}}{e^{-2\pi i \bar{u}} - 1} - 1 \right] \simeq \frac{\pi}{B} \quad \text{for } B \to +\infty \tag{A.5}$$

with $u = A + iB$, $A = U_1^{(3)} s_1$ and $B = U_2^{(3)} s_1$ (or $B = \sqrt{(U_2^{(3)} s_1)^2 + N U_2^{(3)}}$), we have:

$$I = -2 \sum_{s_1 > 0} \left[\frac{1}{s_1} \frac{q^{s_1}}{q^{s_1} - 1} + \frac{1}{s_1} \frac{\bar{q}^{s_1}}{\bar{q}^{s_1} - 1} \right] + \sum_{s_1 > 0} \left[\frac{2}{s_1} - \frac{2}{\sqrt{s_1^2 + \frac{N}{U_2}}} \right]$$

$$+ \frac{U_2^{(3)}}{\pi} \sum_{s_2 \in \mathbb{Z} - \{0\}} \left[\frac{1}{s_2^2} - \frac{1}{s_2^2 + N U_2^{(3)}} \right] \tag{A.6}$$

with $q = e^{2\pi i U^{(3)}}$. Expanding the geometric series, the first term gives

$$-2 \sum_{s_1 > 0} \frac{1}{s_1} \frac{q^{s_1}}{q^{s_1} - 1} = 2 \sum_{n, s_1 > 0} \frac{1}{s_1} q^{n s_1} - \log(q^{-1/6} \eta(U^{(3)})^2) \,, \tag{A.7}$$

where $\eta(U)$ is the Dedekind η-function. The second term can be evaluated by using the Euler-Maclaurin formula:

$$2 \sum_{s_1 > 0} \left[\frac{1}{s_1} - \frac{1}{\sqrt{s_1^2 + \frac{N}{U_2^{(3)}}}} \right] \simeq 2 \log \frac{\sqrt{N}}{2\sqrt{U_2^{(3)}}} + 2\gamma_E \,. \tag{A.8}$$

The last term yields:

$$\frac{U_2^{(3)}}{\pi} \sum_{s_2 \in \mathbb{Z} - \{0\}} \left[\frac{1}{s_2^2} - \frac{1}{s_2^2 + N} \right] = 2 \frac{U_2^{(3)}}{\pi} \zeta(2) - O(m^2) \simeq + \frac{\pi}{3} U_2^{(3)} \,, \tag{A.9}$$

where we used the particular value of Riemann zeta function $\zeta(2) = \pi^2/6$. Finally we can write:

$$I = -\log |\eta(U^{(3)})|^4 - \log \left(U_2^{(3)} T_2^{(3)} |\ell_a^{(3)}|^2 \right) - \log \left(\alpha' m^2 \right) \tag{A.10}$$

where we have redefined $2m^2 e^{-2\gamma_E} \to m^2$.

Regularization with Wilson lines We now briefly describe the effect of introducing Wilson lines on the torus $\mathcal{T}_2^{(3)}$ which can act as IR regulators [26] for the integral I in Eq. (A.1).

Turning on Wilson lines ξ_1 and ξ_2 along $\mathcal{T}_2^{(3)}$ produces a shift on the momenta so that I becomes

$$K(\xi_1, \xi_2) \equiv \int_0^\infty \frac{d\tau}{\tau} \sum_{(r_1, r_2) \in \mathbb{Z}^2} e^{-2\pi\tau \frac{|(r_1 - \xi_1)U^{(3)} - (r_2 - \xi_2)|^2}{U_2^{(3)} T_2^{(3)} |\ell_a^{(3)}|^2}} \,. \tag{A.11}$$

Subtracting the UV divergence after a Poisson resummation as we did before in Eq. (A.3), we have

$$K(\xi_1, \xi_2) \equiv \frac{|\ell_a^{(3)}|^2 T_2^{(3)}}{2} \int_0^\infty \frac{d\tau}{\tau^2} \sum_{(s_1, s_2) \in \mathbb{Z}^2 - \{(0,0)\}} e^{-\frac{\pi}{2\tau} \frac{|\ell_a^{(3)}|^2 T^{(3)}}{U_2^{(3)}} |U^{(3)} s_1 + s_2|^2 + 2\pi i (s_1 \xi_1 + s_2 \xi_2)}$$

$$\tag{A.12}$$

which can be easily integrated to give

$$K(\xi_1, \xi_2) = \frac{U_2^{(3)}}{\pi} \sum_{(s_1, s_2) \in \mathbb{Z}^2 - \{(0,0)\}} \frac{e^{2\pi i (s_1 \xi_1 + s_2 \xi_2)}}{|U^{(3)} s_1 + s_2|^2} \,. \tag{A.13}$$

If $\xi_2 = 0$ we can use Eq. (A.5) and write

$$K(\xi_1, \xi_2 = 0) = - \sum_{s_1 > 0} \left[\frac{1}{s_1} \frac{q^{s_1}(e^{2\pi i \xi_1 s_1} + e^{-2\pi i \xi_1 s_1})}{q^{s_1} - 1} + \frac{1}{s_1} \frac{\bar{q}^{s_1}(e^{2\pi i \xi_1 s_1} + e^{-2\pi i \xi_1 s_1})}{\bar{q}^{s_1} - 1} \right]$$

$$+ \sum_{s_1 > 0} \frac{(e^{2\pi i \xi_1 s_1} + e^{-2\pi i \xi_1 s_1})}{s_1} + \frac{U_2^{(3)}}{\pi} \sum_{s_2 \in \mathbb{Z} - \{0\}} \frac{1}{s_2^2} \tag{A.14}$$

with $q = e^{2\pi i U^{(3)}}$. Expanding the geometric series, the first term gives

$$-\sum_{s_1 > 0} \frac{1}{s_1} \frac{q^{s_1}(e^{2\pi i \xi_1 s_1} + e^{-2\pi i \xi_1 s_1})}{q^{s_1} - 1} = -\sum_{n>0} \log\left((1 - e^{2\pi i \xi_1} q^n)(1 - e^{-2\pi i \xi_1} q^n)\right) , \quad (A.15)$$

and similarly for the second term with q replaced by $\bar{q}$. The second line of (A.14) can be easily seen to give

$$-\log\left(4\sin^2(\pi\xi_1)\right) + \frac{\pi}{3} U_2^{(3)} , \quad (A.16)$$

so that we can finally write

$$K(\xi_1, \xi_2 = 0) = \frac{\pi}{3} U_2^{(3)} - \log\left|2\sin(\pi\xi_1)\prod_{n=1}^{\infty}(1 - e^{2\pi i \xi_1} q^n)(1 - q^n e^{-2\pi i \xi_1})\right|^2$$

$$= -\log\left|\frac{\theta_1(\xi_1| - i U^{(3)})}{\eta(U^{(3)})}\right|^2 . \quad (A.17)$$

To find the general expression for $\xi_2 \neq 0$, it is convenient to introduce the complex variable $z = \xi_1 - U^{(3)}\xi_2$ such that $z \simeq z + 1 \simeq z - U^{(3)}$ as a consequence of the periodicity of the Wilson lines. Then, one can show that

$$\frac{\partial}{\partial\bar{z}}\frac{\partial}{\partial z} K(\xi_1, \xi_2) = \frac{\pi}{U_2^{(3)}}\left[1 - \delta(\xi_1)\,\delta(\xi_2)\right] . \quad (A.18)$$

Studying the behavior of the solution to this differential equation near $z = 0$ and matching with the form (A.17) of the explicit solution already found for $\xi_2 = 0$, one can obtain [26, 38]

$$K(\xi_1, \xi_2) = -\log\left|e^{-i\pi\xi_2 U^{(3)}} \frac{\theta_1(z| - i U^{(3)})}{\eta(U^{(3)})}\right|^2 . \quad (A.19)$$

This final result can be entirely written as the sum of a holomorphic and an anti-holomorphic function, in agreement with the fact that in the Wilson line regularization all excitations are massive.

References

[1] J. Polchinski, Phys. Rev. D **50** (1994) 6041 [arXiv:hep-th/9407031].

[2] K. Becker, M. Becker and A. Strominger, Nucl. Phys. B **456** (1995) 130 [arXiv:hep-th/9507158].

[3] M. B. Green and P. Vanhove, Phys. Lett. B **408** (1997) 122 [arXiv:hep-th/9704145].

[4] M. B. Green, M. Gutperle and P. Vanhove, Phys. Lett. B **409** (1997) 177 [arXiv:hep-th/9706175].

[5] E. Kiritsis and B. Pioline, Nucl. Phys. B **508** (1997) 509 [arXiv:hep-th/9707018].

[6] C. Bachas, C. Fabre, E. Kiritsis, N. A. Obers and P. Vanhove, Nucl. Phys. B **509** (1998) 33 [arXiv:hep-th/9707126].

[7] M.B. Green and M. Gutperle, Nucl.Phys. B **498** (1997) 195 [arXiv:hep-th/9701093]; JHEP **9801** (1998) 005 [arXiv:hep-th/9711107]; Phys. Rev. D **58** (1998) 046007 [arXiv:hep-th/9804123].

[8] M.B. Green and M. Gutperle, JHEP **0002** (2000) 014 [arXiv:hep-th/0002011].

[9] T. Banks and M. B. Green, JHEP **9805** (1998) 002 [arXiv:hep-th/9804170].

[10] C. S. Chu, P. M. Ho and Y. Y. Wu, Nucl. Phys. B **541**, 179 (1999) [arXiv:hep-th/9806103].

[11] I. I. Kogan and G. Luzon, Nucl. Phys. B **539** (1999) 121 [arXiv:hep-th/9806197].

[12] M. Bianchi, M. B. Green, S. Kovacs and G. Rossi, JHEP **9808** (1998) 013 [arXiv:hep-th/9807033].

[13] N. Dorey, V. V. Khoze, M. P. Mattis and S. Vandoren, Phys. Lett. B **442** (1998) 145 [arXiv:hep-th/9808157]; N. Dorey, T. J. Hollowood, V. V. Khoze, M. P. Mattis and S. Vandoren, JHEP **9906** (1999) 023 [arXiv:hep-th/9810243]; Nucl. Phys. B **552** (1999) 88 [arXiv:hep-th/9901128].

[14] E. Witten, Nucl. Phys. B **460** (1996) 335 [arXiv:hep-th/9510135].

[15] M. R. Douglas, J. Geom. Phys. **28** (1998) 255 [arXiv:hep-th/9604198]. arXiv:hep-th/9512077.

[16] N. Dorey, T. J. Hollowood, V. V. Khoze and M. P. Mattis, Phys. Rept. **371** (2002) 231 [arXiv:hep-th/0206063].

[17] M. Bianchi, S. Kovacs and G. Rossi, arXiv:hep-th/0703142.

[18] M. Billo, M. Frau, I. Pesando, F. Fucito, A. Lerda and A. Liccardo, JHEP **0302** (2003) 045 [arXiv:hep-th/0211250].

[19] P. Di Vecchia, M. Frau, I. Pesando, S. Sciuto, A. Lerda and R. Russo, Nucl. Phys. B **507** (1997) 259 [arXiv:hep-th/9707068].

[20] M. Billo, M. Frau, I. Pesando and A. Lerda, JHEP **0405** (2004) 023 [arXiv:hep-th/0402160].

[21] M. Billo, M. Frau, S. Sciuto, G. Vallone and A. Lerda, JHEP **0605** (2006) 069 [arXiv:hep-th/0511036].

[22] M. Billo, M. Frau, F. Fucito and A. Lerda, JHEP **0611** (2006) 012 [arXiv:hep-th/0606013].

[23] C. Beasley and E. Witten, JHEP **0602** (2006) 060 [arXiv:hep-th/0512039].

[24] R. Blumenhagen, M. Cvetic and T. Weigand, Nucl. Phys. B **771** (2007) 113 [arXiv:hep-th/0609191].

[25] L. E. Ibanez and A. M. Uranga, JHEP **0703** (2007) 052 [arXiv:hep-th/0609213].

[26] S. A. Abel and M. D. Goodsell, arXiv:hep-th/0612110.

[27] N. Akerblom, R. Blumenhagen, D. Lüst, E. Plauschinn and M. Schmidt-Sommerfeld, JHEP **0704** (2007) 076 [arXiv:hep-th/0612132].

[28] M. Bianchi and E. Kiritsis, arXiv:hep-th/0702015.

[29] M. Cvetic, R. Richter and T. Weigand, arXiv:hep-th/0703028.

[30] R. Argurio, M. Bertolini, G. Ferretti, A. Lerda and C. Petersson, JHEP **0706** (2007) 067 [arXiv:0704.0262 [hep-th]].

[31] M. Bianchi, F. Fucito and J. F. Morales, arXiv:0704.0784 [hep-th].

[32] L. E. Ibanez, A. N. Schellekens and A. M. Uranga, arXiv:0704.1079 [hep-th].

[33] N. Akerblom, R. Blumenhagen, D. Lust and M. Schmidt-Sommerfeld, arXiv:0705.2366 [hep-th].

[34] S. Antusch, L. E. Ibanez and T. Macri, arXiv:0706.2132 [hep-ph].

[35] R. Blumenhagen, M. Cvetic, D. Lust, R. Richter and T. Weigand, arXiv:0707.1871 [hep-th].

[36] O. Aharony and S. Kachru, arXiv:0707.3126 [hep-th].

[37] R. Blumenhagen, M. Cvetic, R. Richter and T. Weigand, arXiv:0708.0403 [hep-th].

[38] R. Blumenhagen, B. Kors, D. Lust and S. Stieberger, arXiv:hep-th/0610327.

[39] V. Kaplunovsky and J. Louis, Nucl. Phys. B **422** (1994) 57 [arXiv:hep-th/9402005]. Nucl. Phys. B **444** (1995) 191 [arXiv:hep-th/9502077].

[40] J. Louis and K. Foerger, Nucl. Phys. Proc. Suppl. **55B** (1997) 33 [arXiv:hep-th/9611184].

[41] M. A. Shifman and A. I. Vainshtein, Nucl. Phys. B **277** (1986) 456.

[42] J. P. Derendinger, S. Ferrara, C. Kounnas and F. Zwirner, Phys. Lett. B **271** (1991) 307; Nucl. Phys. B **372** (1992) 145.

[43] G. Lopes Cardoso and B. A. Ovrut, Nucl. Phys. B **392** (1993) 315 [arXiv:hep-th/9205009].

[44] L. E. Ibanez and D. Lust, Nucl. Phys. B **382** (1992) 305 [arXiv:hep-th/9202046].

[45] M. K. Gaillard and T. R. Taylor, Nucl. Phys. B **381** (1992) 577 [arXiv:hep-th/9202059].

[46] P. Binetruy, G. Girardi and R. Grimm, Phys. Lett. B **265**, 111 (1991).

[47] B. de Wit, V. Kaplunovsky, J. Louis and D. Lust, Nucl. Phys. B **451** (1995) 53 [arXiv:hep-th/9504006].

[48] T. J. Hollowood, JHEP **0203** (2002) 038 [arXiv:hep-th/0201075].

[49] D. Lüst and S. Stieberger, arXiv:hep-th/0302221.

[50] M. R. Douglas and M. Li, arXiv:hep-th/9604041.

[51] C. Bachas and C. Fabre, Nucl. Phys. B **476** (1996) 418 [arXiv:hep-th/9605028].

[52] P. Di Vecchia, A. Liccardo, R. Marotta and F. Pezzella, Int. J. Mod. Phys. A **20** (2005) 4699 [arXiv:hep-th/0503156].

[53] D. Lüst, P. Mayr, R. Richter and S. Stieberger, Nucl. Phys. B **696** (2004) 205 [arXiv:hep-th/0404134].

[54] I. Antoniadis, C. Bachas, C. Fabre, H. Partouche and T. R. Taylor, Nucl. Phys. B **489** (1997) 160 [arXiv:hep-th/9608012].

[55] M. Berg, M. Haack and B. Körs, JHEP **0511** (2005) 030 [arXiv:hep-th/0508043].

[56] M. Bertolini, M. Billo, A. Lerda, J. F. Morales and R. Russo, Nucl. Phys. B **743** (2006) 1 [arXiv:hep-th/0512067].

[57] N. A. Nekrasov, Adv. Theor. Math. Phys. **7** (2004) 831 [arXiv:hep-th/0206161]; R. Flume and R. Poghossian, Int. J. Mod. Phys. A **18** (2003) 2541 [arXiv:hep-th/0208176]; R. Flume, F. Fucito, J. F. Morales and R. Poghossian, JHEP **0404** (2004) 008 [arXiv:hep-th/0403057].

[58] P. Di Vecchia, A. Liccardo, R. Marotta and F. Pezzella, JHEP **0306** (2003) 007 [arXiv:hep-th/0305061].

[59] A. D'Adda and P. Di Vecchia, Phys. Lett. B **73** (1978) 162.

[60] M. Billo, P. Di Vecchia, M. Frau, A. Lerda, I. Pesando, R. Russo and S. Sciuto, Nucl. Phys. B **526** (1998) 199 [arXiv:hep-th/9802088].

[61] M. Billo, B. Craps and F. Roose, JHEP **0101** (2001) 038 [arXiv:hep-th/0011060].

[62] M. Billo, P. Di Vecchia, M. Frau, A. Lerda, R. Marotta and I. Pesando, arXiv:0709.0245 [hep-th].

[63] L. J. Dixon, V. Kaplunovsky and J. Louis, Nucl. Phys. B **355** (1991) 649.